Some odd finitely presented groups

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1. Introduction

Once upon a time we set out to construct some truly exotic finitely presented groups. As it turned out we didn't succeed in finding the groups we wanted, but we nevertheless managed to concoct some groups with rather interesting properties. The purpose of this note is to record these examples and constructions.

If G is a group, an element $g \in G$ is said to be *divisible* if for every natural number $n > 0$ there is an $x \in G$ such that $x^n = g$. The group G itself is called *divisible* if every element in G is divisible. As a consequence of the theory of amalgamated free products B. H. Neumann[8] showed that any group G can be embedded in a suitable divisible group, say $D(G)$. Now one can ask whether there are finitely generated, or even finitely presented, divisible groups. Rips[9] has announced the construction of a finitely generated divisible group using his strengthened versions of small cancellation theory, but proofs have not yet appeared. Our first concoction is the following:

Theorem A. There exists a finitely presented group G whose commutator subgroup $[G, G]$ is divisible and infinite and $G/[G, G] \cong \mathbf{Z}$.

Using this we can also show the following:

Theorem B. Every finitely presented group can be embedded in a finitely presented group M which is generated by four divisible subgroups. Alternatively, M is generated by eight divisible elements.

Our second class of examples relate to groups which are isomorphic to their direct squares. In [7] B. H. Neumann constructed a finitely generated group which has each of the alternating groups A_n for n odd as a direct factor. Then in 1970 J. M. Tyrer Jones [10] constructed a finitely generated group G such that $G \cong G \times G$. More recently Meier[6] has constructed more in a systematic way. His constructions have led us to the following:

Theorem C. There is a non-trivial finitely presented group H such that its direct square $H \times H$ is a quotient group of H, i.e. there is an epimorphism $\phi : H \to H \times H$.

Of course such a group must be non-hopfian, but for rather different reasons from the usual examples. The group H we construct is generated by four elements, so it follows that the rank (i.e., minimum number of generators) of any finite direct power $H \times H \times ...H$ of $H$is at most four.

The group of Theorem C in turn leads easily to the following example:

Theorem D. There is a finitely presented group G which is an HNN-extension with two stable letters of a non-trivial finitely generated group K such that K is isomorphic to $K \times K$.

In variations on the last two results we can arrange matters so that the groups in question contain an isomorphic copy of every finitely presented group.

2. Divisible groups

Our proof of Theorem A depends on a trick similar to one we introduced in [1] to embed finitely presented groups into finitely presented acyclic groups. It follows from the Higman Embedding Theorem that there is a finitely presented group U which contains an isomorphic copy of every finitely presented group, and indeed every recursively presented group (see [4] or [5]). Now U can be chosen to be a two generator group so we can write its presentation as say

$$U = \langle x, y | r_1 = 1, ..., r_m = 1 \rangle .$$

Then U can be embedded in a divisible group $D(U)$ by an inductive construction due to B. H. Neumann [8] as follows: let $D_0 = U$. Assume that D_i has been constructed and let $w_1, w_2, ...$ be an enumeration of all words in the generators of D_i. Form the presentation for D_{i+1} from that of D_i by adding new generating symbols c_{jn} for all $j, n > 0$ together with the additional defining relations $w_j = c_{jn}^n$ for all $j, n > 0$. Finally take $D(U)$ to be the union of these presentations. It follows from the theory of amalgamated free products that U is embedded in $D(U)$ which is clearly divisible and recursively presented. Hence $D(U)$ can in turn be embedded in U, so we have embeddings $U \to D(U) \to U$.

Changing our viewpoint slightly, we know that U contains subgroups $U_0 \subset D \subset U$ such that D is divisible and U is isomorphic to U_0 say via an isomorphism $\theta : U \to U_0$. Now form the HNN-extension

$$G = \left\langle U, t | t^{-1}xt = \theta(x), t^{-1}yt = \theta(y) \right\rangle .$$

In G the element t conjugates U into its subgroup U_0. Thus the normal closure of U is the ascending union of subgroups

$$U_0 = t^{-1}Ut \subset U \subset tUt^{-1} \subset t^2Ut^{-2} \subset .....$$

Since U_0 is contained in a divisible subgroup of U, it follows that the union of this chain is a divisible group. Moreover it is the commutator subgroup $[G, G]$ of G and $G/[G, G] \cong \langle t | \ \rangle$, the infinite cyclic group generated by t. This completes the proof of Theorem A.

Before beginning the proof of Theorem B, we record the following frequently used observation: Suppose that $G = H *_A K$ is the free product of groups H and K with amalgamated subgroup A. If H_1 and K_1 are subgroups of H and K respectively such that $H_1 \cap A = 1 = K_1 \cap A$ then the subgroup of G generated by H_1 and K_1 is their ordinary free product. This is an elementary consequence of the normal form theorem for amalgamated free products, although it also follows from their subgroup theory.

We are now ready to begin the proof of Theorem B for which we use a variation on a theme of Higman [3]. Let $G_i, i = 1, .., 4$ be isomorphic copies of the group G constructed above in proving Theorem A. Select an element $z \in U$ having infinite order. If w is an element of G, we denote by w_i the corresponding element of G_i. Next form the amalgamated free product $L = \langle G_1 * G_2 | t_1 = z_2 \rangle$. Then by our previous observation, z_1 and t_2 freely generate a free subgroup of rank two in L. Similarly, let $R = \langle G_3 * G_4 | t_3 = z_4 \rangle$ in which the subgroup generated by z_3 and t_4 is free of rank two. Finally form the amalgamated free product of L and R, reversing top and bottom as in [3] to obtain:

$$M = \langle L * R | t_2 = z_3, z_1 = t_4 \rangle .$$

Now one can easily check that M is generated by $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$. Moreover each of these generators lies in one of the four divisible subgroups $[G_i, G_i]$ of M, thus establishing Theorem B.

3. Groups mapping onto their direct square.

Before beginning the proofs of Theorems C and D we record a few easy observations some of which are due to P. M. Neumann.

Propostion 1. Let G be a non-trivial group which is isomorphic to its direct square $G \times G$. Then G contains a normal subgroup N which is isomorphic to the direct sum of infinitely many copies of G. Moreover,the centralizer of any element of G contains an abelian subgroup of infinite rank.

Proof: Write $G = G_0 \times G_1$ where the subgroups G_i are each isomorphic to G. Now in turn each $G_i = G_{i0} \times G_{i1}$ where again each of the subgroups G_{ij} is isomorphic to G. Continuing in this manner, it follows that at stage n the group G can be written as a direct product of 2^n of its subgroups, each of which is isomorphic to G. These subgroups can be arranged as the vertices on a binary branching tree with each vertex at level n indexed by a sequence of n 0's and 1's. We call such a sequence a *binary sequence of length n.*

First, we observe that

$$G = G_0 \times G_1 = G_0 \times G_{10} \times G_{11} = G_0 \times G_{10} \times G_{110} \times \ldots \times G_{11\ldots10} \times G_{11\ldots11}.$$

Letting N be the subgroup generated by the set of subgroups $\{G_\alpha | \alpha = 111\ldots110\}$ it follows that N is in fact the direct sum of these G_α and is a normal subgroup. This proves the first assertion. Clearly then G contains an abelian subgroup of infinite rank.

If $x \in G$ then $x = x_0x_1 = x_{00}x_{01}x_{10}x_{11} = \ldots etc.$ where for any binary sequence α we denote by x_α the projection of x onto the corresponding vertex G_α. Certainly x_α commutes with x.

Suppose now that some $x_\alpha = 1$. Then the corresponding G_α is contained in the centralizer of x. But G_α is isomorphic to G and so contains an abelian subgroup of infinite rank. Finally, suppose none of the x_α are trivial. One shows easily that they must generate an abelian subgroup of infinite rank. This completes the proof of the proposition.

Propostion 2. Let H be a non-trivial finitely generated group such that $H \times H$ is a quotient group of H, say via the epimorphism $\phi : H \to H \times H$. Then H is a perfect, non-hopfian group which contains no proper subgroups of finite index. Also rank $(H \times H \times \ldots \times H) \leq$ rank(H) for any finite direct power of H.

Proof: All of the assertions are clear except perhaps the non-existence of subgroups of finite index. Suppose H had a proper subgroup of finite index n. Since H is finitely generated, it can have at most finitely many such subgroups of index n, say k of them. But then $H \times H$ has at least $2k$ subgroups of index n, and their pre-images under ϕ are at least $2k$ subgroups of H of index n which is a contradiction, completing the proof.

Now we turn to the proof of Theorem C. We use the common notation for commutators $[x, y] = x^{-1}y^{-1}xy$ and for exponentiation $x^y = y^{-1}xy$. Consider the following two finitely presented groups:

$$A = \left\langle a, h, t \mid [a, h] = 1, a^{2t} = a^3, h^{2t} = h^3 \right\rangle$$

$$B = \left\langle b, s \mid b^{2s} = b^3 \right\rangle.$$

Of course B is just the familiar non-hopfian group of Baumslag and Solitar [2] while A is an HNN-extension of the free abelian group with basis $\{a, h\}$ by a stable letter t conjugating the square of each to its cube. A is also non-hopfian, but we will not need to know that either A or B is non-hopfian.

Now in A the elements t and $[a, a^t]$ freely generate a free subgroup, and similarly in B the elements s and $[b, b^s]$ freely generate a free subgroup. Hence the free product with amalgamation

$$\begin{aligned} H &= \left\langle A * B \mid t = [b, b^s], [a, a^t] = s \right\rangle \\ &= \left\langle a, h, b, s, t \mid [a, h] = 1, a^{2t} = a^3, h^{2t} = h^3, b^{2s} = b^3, t = [b, b^s], [a, a^t] = s \right\rangle \end{aligned}$$

embeds both A and B, and is non-trivial in particular.

Let H_0 and H_1 be isomorphic copies of H. The elements of these groups corresponding to a, etc. will be denoted a_0 and a_1, etc. Let $D = H_0 \times H_1$ which is isomorphic to the direct square of H. Define a function ϕ from the generators of H to D as follows:

$$\phi(a) = a_0a_1, \phi(b) = b_0b_1, \phi(s) = s_0s_1, \phi(t) = t_0t_1, \phi(h) = a_0h_0^2h_1^2.$$

Firstly, we claim that the formal extension of ϕ to all words on the generators of H defines a homomorphism from H to D. It is clear that the relations not involving h are preserved, so it suffices to check that these too are preserved. This follows from the following equations in D:

$$(a_0h_0^2h_1^2)^{2t_0t_1} = a_0^{2t_0}h_0^{4t_0}h_1^{4t_1} = a_0^3h_0^6h_1^6 = (a_0h_0^2h_1^2)^3$$

and

$$[a_0a_1, a_0h_0^2h_1^2] = [a_0, a_0h_0^2][a_1, h_1^2] = 1.$$

Next consider the subgroup C of D generated by the elements $\{a_0a_1, b_0b_1, s_0s_1, t_0t_1, a_0h_0^2h_1^2\}$ which are the images under ϕ of the generators of H. We claim that $C = D$ so that ϕ is a homomorphism from H onto D which will establish Theorem C. We use the relations of D to show that each of the generators of D lies in C. First

$$[a_0a_1, (a_0h_0^2h_1^2)^{t_0t_1}] = [a_0, (a_0h_0^2)^{t_0}][a_1, h_1^{2t_1}] = [a_0, a_0^{t_0}] = s_0 \in C.$$

So $s_0 \in C$, whence $s_1 \in C$. Then

$$[b_0b_1, (b_0b_1)^{s_0}] = [b_0, b_0^{s_0}] = t_0 \in C$$

and hence $t_1 \in C$. Also

$$[(a_0a_1)^2, t_0] = [a_0^2, t_0] = a_0 \in C$$

and so $a_1 \in C$. Similarly equations give $b_0, b_1 \in C$. Since $a_0 \in C$, we have $h_0^2h_1^2 \in C$ and so

$$[h_0^2h_1^2, t_0] = [h_0^2, t_0] = h_0 \in C$$

and similarly $h_1 \in C$. Thus each generator of C lies in D and $C = D$, completing the proof of Theorem C.

Continuing the above notation, we observe there are other epimorphisms from H onto $H_0 \times H_1$. One we make use of below can be obtained by observing that the map from H to itself defined by sending $h \mapsto a^{-1}h$ and fixing the other generators is an automorphism of H. Composing this with the epimorphism ϕ above gives the map θ defined by

$$\theta(a) = a_0a_1, \theta(b) = b_0b_1, \theta(s) = s_0s_1, \theta(t) = t_0t_1, \theta(h) = a_1^{-1}h_0^2h_1^2$$

which is another epimorphism from H onto $H_0 \times H_1$. We chose to use ϕ above instead of θ because the calculations are more pleasant.

To prove Theorem D we start with two copies H_0 and H_1 as above of the group H constructed in the proof of Theorem C. Let λ be the automorphism of $H_0 \times H_1$ which interchanges the isomorphic direct factors. Now form a new group G by taking a presentation for $H_0 \times H_1$ and adding two new generators p and q and the following equations :

$$p^{-1}xp = \lambda(x) \text{ for } x \text{ among the given generators of } H_0 \text{ and } H_1$$

$$q^{-1}a_0q = a_0a_1, q^{-1}b_0q = b_0b_1, q^{-1}s_0q = s_0s_1, q^{-1}t_0q = t_0t_1, q^{-1}h_0q = a_1^{-1}h_0^2h_1^2.$$

Because the map θ discussed above is not an isomorphism, H_0 and H_1 are not embedded in G. Let K_0 and K_1 denote the images of H_0 and H_1 in G and let K be the subgroup they generate, which is of course their direct product. Now K_0 and K_1 are isomorphic subgroups of G because they are conjugate by p. Since the map θ is an epimorphism, the element q conjugates K_0 onto K. Thus K_0, K_1 and K are all isomorphic and $K = K_0 \times K_1$. Moreover, it is clear that G is an HNN-extension of its finitely generated subgroup K with stable letters p and q and that G is finitely presented.

It remains to show that K is non-trivial. To prove this we produce a model M which is a non-trivial quotient group of H such that $M \cong M \times M$ where the isomorphism is given by the map corresponding to the above θ. The required group M was constructed by Meier[6]. We need to briefly review his construction. Consider the finitely presented group

$$T = \left\langle a, b, s, t | a^{2t} = a^3, b^{2s} = b^3, t = [b, b^s], [a, a^t] = s \right\rangle.$$

Now T is the free product of two copies of the Baumslag-Solitar group amalgamating a free subgroup of rank two. Also it is the quotient group of H obtained by setting h equal to the identity.

Now form the direct power $T^{\mathbf{N}}$ where $\mathbf{N}$ is the natural numbers. The elements of $T^{\mathbf{N}}$ will be written as infinite vectors, and we introduce the notation $\mathbf{a} = (a, a, a, a, \ldots)$,

$\mathbf{a}_0 = (1, a, 1, a, \ldots)$, $\mathbf{a}_1 = (a, 1, a, 1, \ldots)$ and similarly for $\mathbf{b}$, $\mathbf{s}$ and $\mathbf{t}$. Also define $\mathbf{h} = (a, a^2, a^3, a^4, \ldots)$ and $\mathbf{h}_0 = (1, a, 1, a^2, \ldots)$ and $\mathbf{h}_1 = (a, 1, a^2, 1, \ldots)$. Let M be the subgroup of $T^{\mathbf{N}}$ generated by $\mathbf{a}, \mathbf{b}, \mathbf{s}, \mathbf{t}$ and $\mathbf{h}$, and similarly define M_0 and M_1. Then M, M_0 and M_1 are clearly all isomorphic and non-trivial, and M_0 and M_1 generate their direct product. But one also verifies that

$$\mathbf{a}_1^{-1}\mathbf{h}_0^2\mathbf{h}_1^2 = (a, a^2, a^3, a^4, \ldots) = \mathbf{h}.$$

Because θ is an epimorphism, it follows $M = M_0 \times M_1$. Moreover, G maps onto the obvious HNN-extension of M in such a way that K maps onto M. Hence K is non-trivial and the proof of Theorem D is complete.

We remark that the above proofs of Theorems C and D taken together in fact prove that the finitely generated group M is isomorphic to its direct square without reference to [6].

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