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Algebraic Topology

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PREFACE

When the present volume was first contemplated some five years ago it was primarily meant to be a second edition of the author's *Topology* (1930, Volume XII of the American Mathematical Society Colloquium Series). It soon became evident however that the subject had moved too rapidly for a mere revised edition, and that a completely new book would have to be written. With the consent of the Colloquium Committee the task was undertaken by the author and resulted in the present work. Its basic topic, often referred to as "Combinatorial Topology," is in substance the theory of complexes and its applications. Many factors have contributed to a great increase in the role of algebra in this subject. For this reason it is more appropriately described as "Algebraic Topology," and this explains the title of the volume.

The purely topological (non-algebraic) part has been concentrated in the first chapter, and all the necessary group-theoretic material in the second, thus resulting in a great economy and simplification in the treatment of many questions, notably duality and intersections. The next three chapters deal with the theory of complexes proper. The basic type selected is A. W. Tucker's modified in that the elements may also take negative dimensions. As is well known one of the important recent advances has been the extension to complexes of the duality and intersection properties of manifolds. This may be accomplished by means of special "dual" cycles (the "pseudocycles" of *Topology*, Chapter VI), or by a special dual complex as done by Tucker (companion algebraic development by W. Mayer), or else again with Alexander and Whitney without new elements but with a new boundary operator for the chains. By utilizing negative dimensions it has been possible to associate with each complex X a dual complex X^* such that the relation between the two is wholly symmetrical. As a consequence the "co-theory" of X (Whitney's terminology) appears as the ordinary theory of X^*, and all the duality and intersection properties are obtained by combining the X, X^* relationship with group-duality and group-multiplication in the sense of Pontrjagin. There emerges thus a theory of complexes of purely algebraic nature, with manifolds relegated to the second plane.

The homology theory of topological spaces is taken up in Chapter VII, the necessary limiting processes constituting the theory of nets and webs being dealt with in Chapter VI. We have chosen as our basic theory the Čech homology theory and in substance reduced to it the other known theories thus unifying a domain which has definitely stood in need of it for some time.

The relative concepts which played such an important role in the previous volume have not been neglected in the present. They appear chiefly in the guise of certain binary dissections which run right through complexes, nets and topological spaces, and are at the root of the mechanism of webs.

The last chapter contains the applications to polyhedra and certain related questions, notably a very concise and very general treatment of fixed points. The book concludes with an appendix by Eilenberg and MacLane on the homology groups of infinite complexes and another by Paul Smith on his theory of fixed points of periodic transformations.

Owing to limitations of time and space it has not been possible to take up the applications of algebraic topology. However with Marston Morse's *Calculus of Variations in the Large* (1934, Volume XVIII of the Colloquium Series), W. V. D. Hodge's *The Theory and Applications of Harmonic Integrals* (1941, Cambridge University Press), and a forthcoming volume by Hassler Whitney on sphere spaces, the reader interested in the applications will readily satisfy his curiosity.

Certain deviations from standard usage have been adopted in the text and should be kept in mind. Thus "compact" replaces "bicompact," and "complex" replaces "abstract complex." (A nomenclature of complexes and manifolds is given at the end of Chapter VIII.) All groups are topological (the topology may be discrete); unless otherwise stated homomorphisms are supposed to be continuous and group-isomorphisms topological, exceptions being indicated by the mention "in the algebraic sense." For vector spaces over a field there is a special set of conventions indicated in Chapter II (22.2).

The literature in topology has grown to such proportions that it has been impossible to provide more than a scanty bibliography. References are given by the author's name followed by an appropriate letter in square brackets. Those to the present volume are of the form (IV, 16.3), where IV stands for Chapter IV and 16.3 for the numbering in the chapter.

It has been my good fortune to have obtained sympathetic cooperation and advice from many sources. In preparation of the manuscript invaluable assistance was received from Samuel Eilenberg, W. W. Flexner, N. E. Steenrod, John Tukey, and as regards the second chapter, Claude Chevalley practically acted as a collaborator. Parts of the manuscript in more or less final form or important parts of the proofs were carefully read by Hubert Arnold, E. G. Begle, Paco Lagerstrom, Saunders MacLane, Moses Richardson, Seymour Sherman, J. D. Tamarkin, A. D. Wallace and Hassler Whitney. To one and all it is a great pleasure to express here my appreciation and thanks.

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