Section 3

Mechanics of Solids and Fluids

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3.1 MECHANICS OF SOLIDS

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REFERENCES: Beer and Johnston, ‘‘Mechanics for Engineers,’’ McGraw-Hill. Ginsberg and Genin, ‘‘Statics and Dynamics,’’ Wiley. Higdon and Stiles, ‘‘Engineering Mechanics,’’ Prentice-Hall. Holowenko, ‘‘Dynamics of Machinery,’’ Wiley. Housnor and Hudson, ‘‘Applied Mechanics,’’ Van Nostrand. Meriam, ‘‘Statics and Dynamics,’’ Wiley. Mabie and Ocvirk, ‘‘Mechanisms and Dynamics of Machinery,’’ Wiley. Synge and Griffith, ‘‘Principles of Mechanics,’’ McGraw-Hill. Timoshenko and Young, ‘‘Advanced Dynamics,’’ McGraw-Hill. Timoshenko and Young, ‘‘Engineering Mechanics,’’ McGraw-Hill.

PHYSICAL MECHANICS

Definitions

Force is the action of one body on another which will cause acceleration of the second body unless acted on by an equal and opposite action counteracting the effect of the first body. It is a **vector** quantity.

Time is a measure of the sequence of events. In newtonian mechanics it is an absolute quantity. In relativistic mechanics it is relative to the *frames of reference* in which the sequence of events is observed. The common unit of time is the second.

Inertia is that property of matter which causes a resistance to any change in the motion of a body.

Mass is a quantitative measure of *inertia*.

Acceleration of Gravity Every object which falls in a vacuum at a given position on the earth’s surface will have the same acceleration g. Accurate values of the acceleration of gravity as measured *relative* to the earth’s surface include the effect of the earth’s rotation and flattening at the poles. The international gravity formula for the acceleration of gravity at the earth’s surface is $g = 32.0881(1 + 0.005288 \sin^2 \phi - 0.0000059 \sin^2 2\phi)$ ft/s², where ϕ is latitude in degrees. For extreme accuracy, the local acceleration of gravity must also be corrected for the presence of large water or land masses and for height above sea level. The absolute acceleration of gravity for a nonrotating earth discounts the effect of the earth’s rotation and is rarely used, except outside the earth’s atmosphere. If g_0 represents the absolute acceleration at sea level, the absolute value at an altitude h is $g = g_0R^2/(R + h)^2$, where R is the radius of the earth, approximately 3,960 mi (6,373 km).

Weight is the resultant force of attraction on the mass of a body due to a gravitational field. On the earth, units of weight are based upon an acceleration of gravity of 32.1740 ft/s² (9.80665 m/s²).

Linear momentum is the product of mass and the linear velocity of a particle and is a vector. The moment of the linear-momentum vector about a fixed axis is the **angular momentum** of the particle about that fixed axis. For a rigid body rotating about a fixed axis, angular momentum is defined as the product of moment of inertia and angular velocity, each measured about the fixed axis.

An increment of **work** is defined as the product of an incremental displacement and the component of the force vector in the direction of the displacement or the component of the displacement vector in the direction of the force. The increment of work done by a couple acting on a body during a rotation of $d\theta$ in the plane of the couple is $dU = M\, d\theta$.

Energy is defined as the capacity of a body to do work by reason of its motion or configuration (see **Work and Energy**).

A **vector** is a directed line segment that has both magnitude and direction. In script or text, a vector is distinguished from a scalar V by a boldface-type $\mathbf{V}$. The magnitude of the scalar is the magnitude of the vector, $V = |\mathbf{V}|$.

A frame of reference is a specified set of geometric conditions to which other locations, motion, and time are referred. In newtonian mechanics, the fixed stars are referred to as the **primary (inertial) frame of reference.** Relativistic mechanics denies the existence of a primary reference frame and holds that all reference frames must be described relative to each other.

SYSTEMS AND UNITS OF MEASUREMENTS

In *absolute systems*, the units of **length, mass,** and **time** are considered fundamental quantities, and all other units including that of **force** are derived.

In *gravitational systems*, the units of **length, force,** and **time** are considered fundamental qualities, and all other units including that of **mass** are derived.

In the SI system of units, the unit of mass is the kilogram (kg) and the unit of length is the metre (m). A force of one newton (N) is derived as the force that will give 1 kilogram an acceleration of 1 m/s².

In the English engineering system of units, the unit of mass is the pound mass (lbm) and the unit of length is the foot (ft). A force of one pound (1 lbf) is the force that gives a pound mass (1 lbm) an acceleration equal to the standard acceleration of gravity on the earth, 32.1740 ft/s² (9.80665 m/s²). A slug is the mass that will be accelerated 1 ft/s² by a force of 1 lbf. Therefore, 1 slug = 32.1740 lbm. When described in the gravitational system, mass is a derived unit, being the constant of proportionality between force and acceleration, as determined by Newton’s second law.

General Laws

NEWTON’S LAWS

I. If a balanced force system acts on a particle at rest, it will remain at rest. If a balanced force system acts on a particle in motion, it will remain in motion in a straight line without acceleration.

II. If an unbalanced force system acts on a particle, it will accelerate in proportion to the magnitude and in the direction of the resultant force.

III. When two particles exert forces on each other, these forces are equal in magnitude, opposite in direction, and collinear.

Fundamental Equation The basic relation between mass, acceleration, and force is contained in Newton’s second law of motion. As applied to a particle of mass, $\mathbf{F} = m\mathbf{a}$, force = mass × acceleration. This equation is a vector equation, since the direction of $\mathbf{F}$ must be the direction of $\mathbf{a}$, as well as having $\mathbf{F}$ equal in magnitude to $m\mathbf{a}$. An alternative form of Newton’s second law states that the resultant force is equal to the time rate of change of momentum, $\mathbf{F} = d(m\mathbf{v})/dt$.

Law of the Conservation of Mass The mass of a body remains unchanged by any ordinary physical or chemical change to which it may be subjected.

Law of the Conservation of Energy The principle of conservation of energy requires that the total mechanical energy of a system remain unchanged if it is subjected only to forces which depend on position or configuration.

Law of the Conservation of Momentum The linear momentum of a system of bodies is unchanged if there is no resultant external force on the system. The angular momentum of a system of bodies about a fixed axis is unchanged if there is no resultant external moment about this axis.

Law of Mutual Attraction (Gravitation) Two particles attract each other with a force F proportional to their masses m_1 and m_2 and inversely proportional to the square of the distance r between them, or $F = km_1m_2/r^2$, in which k is the gravitational constant. The value of the gravitational constant is $k = 6.673 \times 10^{-11}$ m³/kg · s² in SI or absolute units, or $k = 3.44 \times 10^{-8}$ ft⁴ lb⁻¹ s⁻⁴ in engineering gravitational units.

It should be pointed out that the unit of force F in the SI system is the **newton** and is derived, while the unit force in the gravitational system is the **pound-force** and is a fundamental quantity.

EXAMPLE. Each of two solid steel spheres 6 in in diam will weigh 32.0 lb on the earth's surface. This is the force of attraction between the earth and the steel sphere. The force of mutual attraction between the spheres if they are just touching is 0.000000136 lb.

STATICS OF RIGID BODIES

General Considerations

If the forces acting on a rigid body do not produce any acceleration, they must neutralize each other, i.e., form a **system of forces in equilibrium.** Equilibrium is said to be **stable** when the body with the forces acting upon it returns to its original position after being displaced a very small amount from that position; **unstable** when the body tends to move still farther from its original position than the very small displacement; and **neutral** when the forces retain their equilibrium when the body is in its new position.

External and Internal Forces The forces by which the individual particles of a body act on each other are known as internal forces. All other forces are called external forces. If a body is supported by other bodies while subject to the action of forces, deformations and forces will be produced at the points of support or contact and these internal forces will be distributed throughout the body until equilibrium exists and the body is said to be in a state of tension, compression, or shear. The forces exerted by the body on the supports are known as **reactions.** They are equal in magnitude and opposite in direction to the forces with which the supports act on the body, known as **supporting forces.** The supporting forces are external forces applied to the body.

In considering a body at a definite section, it will be found that all the internal forces act in pairs, the two forces being equal and opposite. The external forces act singly.

General Law When a body is at rest, the forces acting externally to it must form an equilibrium system. This law will hold for any part of the body, in which case the forces acting at any section of the body become external forces when the part on either side of the section is considered alone. In the case of a **rigid body,** any two forces of the same magnitude, but acting in opposite directions in any straight line, may be added or removed without change in the action of the forces acting on the body, provided the strength of the body is not affected.

Composition, Resolution, and Equilibrium of Forces

The **resultant** of several forces acting at a point is a force which will produce the same effect as all the individual forces acting together.

Forces Acting on a Body at the Same Point The resultant R of two forces F_1 and F_2 applied to a rigid body at the same point is represented in magnitude and direction by the diagonal of the parallelogram formed by F_1 and F_2 (see Figs. 3.1.1 and 3.1.2).

$$R = \sqrt{F_1^2 + F_2^2 + 2\,F_1F_2 \cos a}$$

$$\sin a_1 = (F_2 \sin a)/R \qquad \sin a_2 = (F_1 \sin a)/R$$

When $a = 90°$, $R = \sqrt{F_1^2 + F_2^2}$, $\sin a_1 = F_2/R$, and $\sin a_2 = F_1/R$.

$$\left.\begin{aligned}&\text{When } a = 0°,\ R = F_1 + F_2\\&\text{When } a = 180°,\ R = F_1 - F_2\end{aligned}\right\}\quad \begin{aligned}&\text{Forces act in same}\\&\text{straight line.}\end{aligned}$$

A **force R may be resolved into two component forces** intersecting anywhere on R and acting in the same plane as R, by the reverse of the operation shown by Figs. 3.1.1 and 3.1.2; and by repeating the operation with the components, R may be resolved into any number of component forces intersecting R at the same point and in the same plane.

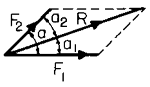


Fig. 3.1.1

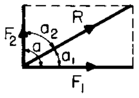


Fig. 3.1.2

Resultant of Any Number of Forces Applied to a Rigid Body at the Same Point Resolve each of the given forces F into components along three rectangular coordinate axes. If A, B, and C are the angles made with XX, YY, and ZZ, respectively, by any force F, the components will be $F \cos A$ along XX, $F \cos B$ along YY, $F \cos C$ along ZZ; add the components of all the forces along each axis algebraically and obtain $\Sigma F \cos A = \Sigma X$ along XX, $\Sigma F \cos B = \Sigma Y$ along YY, and $\Sigma F \cos C = \Sigma Z$ along ZZ.

The resultant $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$. The angles made by the resultant with the three axes are A_r with XX, B_r with YY, C_r with ZZ, where

$$\cos A_r = \Sigma X/R \qquad \cos B_r = \Sigma Y/R \qquad \cos C_r = \Sigma Z/R$$

The **direction** of the **resultant** can be determined by plotting the algebraic sums of the components.

If the forces are all in the same plane, the components of each of the forces along one of the three axes (say ZZ) will be 0; i.e., angle $C_r = 90°$ and $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$, $\cos A_r = \Sigma X/R$, and $\cos B_r = \Sigma Y/R$.

For equilibrium, it is necessary that $R = 0$; i.e., ΣX, ΣY, and ΣZ must each be equal to zero.

General Law In order that a number of forces acting at the same point shall be in equilibrium, the algebraic sum of their components along any *three* coordinate axes must each be equal to zero. When the forces all act in the same plane, the algebraic sum of their components along any *two* coordinate axes must each equal zero.

When the Forces Form a System in Equilibrium Three unknown forces can be determined if the lines of action of the forces are *all* known and are in different planes. If the forces are all in the same plane, the lines of action being known, only *two* unknown forces can be determined. If the lines of action of the unknown forces are *not* known, only *one* unknown force can be determined in either case.

Couples and Moments

Couple Two parallel forces of equal magnitude (Fig. 3.1.3) which act in opposite directions and are not collinear form a couple. **A couple cannot be reduced to a single force.**

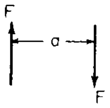


Fig. 3.1.3

Displacement and Change of a Couple The forces forming a couple may be moved about and their magnitude and direction changed, provided they always remain parallel to each other and remain in either the original plane or one parallel to it, and provided the product of one of the forces and the perpendicular distance between the two is constant and the direction of rotation remains the same.

Moment of a Couple The moment of a couple is the product of the magnitude of one of the forces and the perpendicular distance between the lines of action of the forces. Fa = moment of couple; a = arm of couple. If the forces are measured in pounds and the distance a in feet, the **unit of rotation moment** is the foot-pound. If the force is measured in kilograms and the distance in metres, the unit is the metre-kilogram. In the cgs system the unit of rotation moment is 1 cm-dyne.

Rotation moments of couples acting in the same plane are conventionally considered to be positive for counterclockwise moments and negative for clockwise moments, although it is only necessary to be consistent within a given problem. The magnitude, direction, and sense of rotation of a couple are completely determined by its moment axis, or moment vector, which is a line drawn perpendicular to the plane in which the couple acts, with an arrow indicating the direction from which the couple will appear to have right-handed rotation; the length of the line represents the magnitude of the moment of the couple. See

Fig. 3.1.4, in which *AB* represents the magnitude of the moment of the couple. Looking along the line in the direction of the arrow, the couple will have right-handed rotation in any plane perpendicular to the line.

Composition of Couples Couples may be combined by adding their moment vectors geometrically, in accordance with the parallelogram rule, in the same manner in which forces are combined.

Couples lying in the same or parallel planes are added algebraically. Let $+28$ lbf · ft ($+38$ N · m), -42 lbf · ft (-57 N · m), and $+70$ lbf · ft (95 N · m) be the moments of three couples in the same or parallel planes; their resultant is a single couple lying in the same or in a parallel plane, whose moment is $\Sigma M = +28 - 42 + 70 = +56$ lbf · ft ($\Sigma M = +38 - 57 + 95 = 76$ N · m).

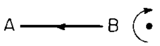


Fig. 3.1.4

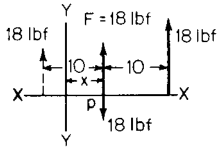


Fig. 3.1.5

If the polygon formed by the moment vectors of several couples closes itself, the couples form an equilibrium system. Two couples will balance each other when they lie in the same or parallel planes and have the same moment in magnitude, but opposite in sign.

Combination of a Couple and a Single Force in the Same Plane (Fig. 3.1.5) Given a force $F = 18$ lbf (80 N) acting as shown at distance *x* from *YY*, and a couple whose moment is -180 lbf · ft (244 N · m) in the same or parallel plane, to find the resultant. A couple may be changed to any other couple in the same or a parallel plane having the same moment and same sign. Let the couple consist of two forces of 18 lbf (80 N) each and let the arm be 10 ft (3.05 m). Place the couple in such a manner that one of its forces is opposed to the given force at *p*. This force of the couple and the given force being of the same magnitude and opposite in direction will neutralize each other, leaving the other force of the couple acting at a distance of 10 ft (3.05 m) from *p* and parallel and equal to the given force 18 lbf (80 N).

General Rule The resultant of a couple and a single force lying in the same or parallel planes is a single force, equal in magnitude, in the same direction and parallel to the single force, and acting at a distance from the line of action of the single force equal to the moment of the couple divided by the single force. The moment of the resultant force about any point on the line of action of the given single force must be of the same sense as that of the couple, positive if the moment of the couple is positive, and negative if the moment of the couple is negative. If the moment of the couple in Fig. 3.1.5 had been + instead of −, the resultant would have been a force of 18 lbf (80 N) acting in the same direction and parallel to *F*, but at a distance of 10 ft (3.05 m) to the left of it (shown dotted), making the moment of the resultant about any point on *F* positive.

To effect a parallel displacement of a single force *F* over a distance *a*, a couple whose moment is *Fa* must be added to the system. The sense of the couple will depend upon which way it is desired to displace force *F*.

The moment of a force with respect to a point is the product of the force *F* and the perpendicular distance from the point to the line of action of the force.

The Moment of a Force with Respect to a Straight Line If the force is resolved into components parallel and perpendicular to the given line, the moment of the force with respect to the line is the product of the magnitude of the perpendicular component and the distance from its line of action to the given line.

Forces with Different Points of Application

Composition of Forces If each force **F** is resolved into components parallel to three rectangular coordinate axes *XX*, *YY*, and *ZZ*, the magnitude of the resultant is $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$, and its line of action makes angles A_r, B_r, and C_r with axes *XX*, *YY*, and *ZZ*, where $\cos A_r = \Sigma X/R$, $\cos B_r = \Sigma Y/R$, and $\cos C_r = \Sigma Z/R$; and there are three couples which may be combined by their moment vectors into a single resultant couple having the moment $M_r = \sqrt{(M_x)^2 + (M_y)^2 + (M_z)^2}$, whose moment vector makes angles of A_m, B_m, and C_m with axes *XX*, *YY*, and *ZZ*, such that $\cos A_m = M_x/M_r$, $\cos B_m = M_y/M_r$, $\cos C_m = M_z/M_r$. If this single resulting couple is in the same plane as the single resulting force at the origin or a plane parallel to it, the system may be reduced to a single force **R** acting at a distance from **R** equal to M_r/R. If the couple and force are not in the same or parallel planes, it is impossible to reduce the system to a single force. If $\mathbf{R} = 0$, i.e., if ΣX, ΣY, and ΣZ all equal zero, the system will reduce to a single couple whose moment is $\mathbf{M}_r$. If $\mathbf{M}_r = 0$, i.e., if M_x, M_y, and M_z all equal zero, the resultant will be a single force **R.**

When the forces are all in the same plane, the cosine of one of the angles A_r, B_r, or $C_r = 0$, say, $C_r = 90°$. Then $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$, $M_r = \sqrt{M_x^2 + M_y^2}$, and the final resultant is a force equal and parallel to *R*, acting at a distance from *R* equal to M_r/R.

A system of forces in the same plane can always be replaced by either a couple or a single force. If $\mathbf{R} = 0$ and $\mathbf{M}_r \cdot 0$, the resultant is a couple. If $\mathbf{M}_r = 0$ and $\mathbf{R} > 0$, the resultant is a single force.

A rigid body is in equilibrium when acted upon by a system of forces whenever $R = 0$ and $\mathrm{M}_r = 0$, i.e., when the following six conditions hold true: $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$, $M_x = 0$, $M_y = 0$, and $M_z = 0$. When the system of forces is in the same plane, equilibrium prevails when the following three conditions hold true: $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma M = 0$.

Forces Applied to Support Rigid Bodies

The external forces in equilibrium acting upon a body may be statically determinate or indeterminate according to the number of unknown forces existing. When the forces are all in the same plane and act at a common point, two unknown forces may be determined if their lines of action are known, one if unknown.

When the forces are all in the same plane and are parallel, two unknown forces may be determined if the lines of action are known, one if unknown.

When the forces are anywhere in the same plane, three unknown forces may be determined if their lines of action are known, if they are not parallel or do not pass through a common point; if the lines of action are unknown, only one unknown force can be determined.

If the forces all act at a common point but are in different planes, three unknown forces can be determined if the lines of action are known, one if unknown.

If the forces act in different planes but are parallel, three unknown forces can be determined if their lines of action are known, one if unknown.

The first step in the solution of problems in statics is the determination of the supporting forces. The following data are required for the complete knowledge of supporting forces: magnitude, direction, and point of application. According to the nature of the problem, none, one, or two of these quantities are known.

One Fixed Support The point of application, direction, and magnitude of the load are known. See Fig. 3.1.6. As the body on which the forces act is in equilibrium, the supporting force *P* must be equal in magnitude and opposite in direction to the resultant of the loads *L*.

In the case of a **rolling surface,** the point of application of the support is obtained from the center of the connecting bolt *A* (Fig. 3.1.7), both the direction and magnitude being unknown. The point of application and

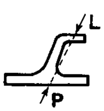


Fig. 3.1.6

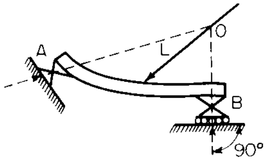


Fig. 3.1.7

line of action of the support at *B* are known, being determined by the rollers.

When three forces acting in the same plane on the same rigid body are in equilibrium, their lines of action must pass through the same point *O*. The load *L* is known in magnitude and direction. The line of action of the support at *B* is known on account of the rollers. The point of application of the support at *A* is known. The three forces are in equilibrium and are in the same plane; therefore, the lines of action must meet at the point *O*.

In the case of the rolling surfaces shown in Fig. 3.1.8, the direction of the support at *A* is known, the magnitude and point of application unknown. The line of action and point of application of the supporting force at *B* are known, its magnitude unknown. The lines of action of the three forces must meet in a point, and the supporting force at *A* must be perpendicular to the plane *XX*. In the case shown in Fig. 3.1.9, the directions and points of application of the supporting forces are known, and the magnitudes unknown. The lines of action of resultant of supports *A* and *B*, the support at *C* and load *L* must meet at a point. Resolve the resultant of supports at *A* and *B* into components at *A* and *B*, their direction being determined by the rollers.

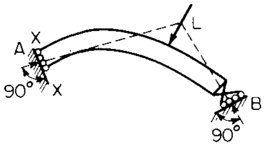


Fig. 3.1.8

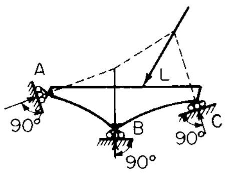


Fig. 3.1.9

If a member of a truss or frame in equilibrium is pinned at two points and loaded at these two points only, the line of action of the forces exerted on the member or by the member at these two points must be along a line connecting the pins.

If the external forces acting upon a rigid body in equilibrium are all in the same plane, the equations $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$ must be satisfied. When trusses, frames, and other structures are under discussion, these equations are usually used as $\Sigma V = 0$, $\Sigma H = 0$, $\Sigma M = 0$, where *V* and *H* represent vertical and horizontal components, respectively.

The **supports** are said to be **statically determinate** when the laws of equilibrium are sufficient for their determination. When the conditions are not sufficient for the determination of the supports or other forces, the structure is said to be **statically indeterminate;** the unknown forces can then be determined from considerations involving the deformation of the material.

When several bodies are so connected to one another as to make up a rigid structure, the forces at the points of connection must be considered as internal forces and are not taken into consideration in the determination of the supporting forces for the structure as a whole.

The distortion of any practically rigid structure under its working loads is so small as to be negligible when determining supporting forces. When the forces acting at the different joints in a built-up structure cannot be determined by dividing the structure up into parts, the structure is said to be **statically indeterminate internally.** A structure may be statically indeterminate internally and still be statically determinate externally.

Fundamental Problems in Graphical Statics

A force may be represented by a straight line in a determined position, and its magnitude by the length of the straight line. The direction in which it acts may be indicated by an arrow.

Polygon of Forces The parallelogram of two forces intersecting each other (see Figs. 3.1.4 and 3.1.5) leads directly to the graphic composition by means of the triangle of forces. In Fig. 3.1.10, *R* is called the **closing side,** and represents the resultant of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in magnitude and direction. Its position is given by the point of application *O*. By means of repeated use of the triangle of forces and by omitting the closing sides of the individual triangles, the magnitude and direction of the resultant **R** of any number of forces in the same plane and intersecting at a single point can be found. In Fig. 3.1.11 the lines representing the forces start from point *O*, and in the force polygon (Fig. 3.1.12) they are joined in any order, the arrows showing their directions following around the polygon in the same direction. The magnitude of the resultant at the point of application of the forces is represented by the closing side **R** of the force polygon; its direction, as shown by the arrow, is counter to that in the other sides of the polygon.

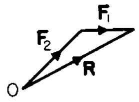


Fig. 3.1.10

If the forces are in equilibrium, R must equal zero, i.e., **the force polygon must close.**

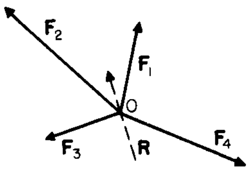


Fig. 3.1.11

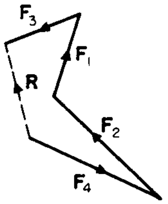


Fig. 3.1.12

If in a closed polygon one of the forces is reversed in direction, this force becomes the resultant of all the others.

If the **forces** do **not** all lie **in the same plane,** the diagram becomes a polygon in space. The resultant **R** of this system may be obtained by adding the forces in space. The resultant is the vector which closes the space polygon. The space polygon may be projected onto three coordinate planes, giving three related plane polygons. Any two of these projections will involve all static equilibrium conditions and will be sufficient for a full description of the force system (see Fig. 3.1.13).

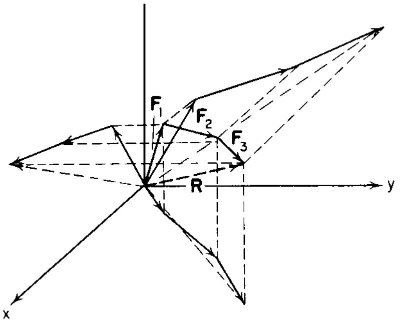


Fig. 3.1.13

Determination of Stresses in Members of a Statically Determinate Plane Structure with Loads at Rest

It will be assumed that the loads are applied at the joints of the structure, i.e., at the points where the different members are connected, and that the connections are pins with no friction. The stresses in the members must then be along lines connecting the pins, unless any member is loaded at more than two points by pin connections. If the members are straight, the forces exerted on them or by them must coincide with the axes of the members, In other words, there shall be no bending stresses in any of the members of the structure.

Equilibrium In order that the whole structure should be in equilibrium, it is necessary that the external forces (loads and supports) shall form a balanced system. Graphical and analytical methods are both of service.

Supporting Forces When the supporting forces are to be determined, it is not necessary to pay any attention to the makeup of the structure under consideration so long as it is practically rigid; the loads may be taken as they occur, or the resultant of the loads may be used instead. When the stresses in the members of the structure are being determined, the loads *must* be distributed at the joints where they belong.

Method of Joints When all the external forces have been determined, any joint at which there are not more than two unknown forces may be taken and these unknown forces determined by the methods of the stress polygon, resolution or moments. In Fig. 3.1.14, let *O* be the joint of a structure and **F** be the only known force; but let *O*1 and *O*2 be two members of the structure joined at *O*. Then the lines of action of the unknown forces are known and their magnitude may be determined (1) by a **stress polygon** which, for equilibrium, must close; (2) by resolution into *H* and *V* components, using the condition of equilibrium $\Sigma H = 0$, $\Sigma V = 0$; or (3) by moments, using any convenient point on the line of action of *O*1 and *O*2 and the condition of equilibrium $\Sigma M = 0$. No more than *two* unknown forces can be determined. In this manner, proceeding from joint to joint, the stresses in all the members of the truss can usually be determined if the structure is statically determinate internally.

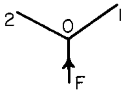


Fig. 3.1.14

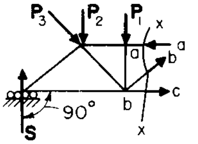


Fig. 3.1.15

Method of Sections The structure may be divided into parts by passing a section through it cutting some of its members; one part may then be treated as a rigid body and the external forces acting upon it determined. Some of these forces will be the stresses in the members themselves. For example, let *xx* (Fig. 3.1.15) be a section taken through a truss loaded at P_1, P_2, and P_3, and supported on rollers at *S*. As the whole truss is in equilibrium, any part of it must be also, and consequently the part shown to the left of *xx* must be in equilibrium under the action of the forces acting externally to it. Three of these forces are the stresses in the members *aa*, *bb*, and *bc*, and are the unknown forces to be determined. They can be determined by applying the condition of equilibrium of forces acting in the same plane but not at the same point. $\Sigma H = 0$, $\Sigma V = 0$, $\Sigma M = 0$. The three unknown forces can be determined only if they are not parallel or do not pass through the same point; if, however, the forces are parallel or meet in a point, two unknown forces only can be determined. Sections may be passed through a structure cutting members in any convenient manner, as a rule, however, cutting not more than three members, unless members are unloaded.

For the determination of stresses in framed structures, see Sec. 12.2.

CENTER OF GRAVITY

Consider a three-dimensional body of any size, shape, and weight. If it is suspended as in Fig. 3.1.16 by a cord from any point *A*, it will be in equilibrium under the action of the tension in the cord and the resultant of the gravity or body forces *W*. If the experiment is repeated by suspending the body from point *B*, it will again be in equilibrium. If the lines of action of the resultant of the body forces were marked in each case, they would be concurrent at a point *G* known as the **center of gravity** or **center of mass.** Whenever the density of the body is uniform, it will be a constant factor and like geometric shapes of different densities will have the same center of gravity. The term **centroid** is used in this case since the location of the center of gravity is of geometric concern only. If densities are nonuniform, like geometric shapes will have the same centroid but different centers of gravity.

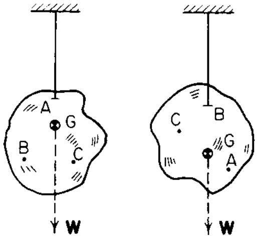


Fig. 3.1.16

Centroids of Technically Important Lines, Areas, and Solids

CENTROIDS OF LINES

Straight Lines The centroid is at its middle point.

Circular Arc *AB* (Fig. 3.1.17*a*) $x_0 = r \sin c/\text{rad } c$; $y_0 = 2r \sin^2 \tfrac{1}{2}c/\text{rad } c$. (rad c = angle c measured in radians.)

Circular Arc *AC* (Fig. 3.1.17*b*) $x_0 = r \sin c/\text{rad } c$; $y_0 = 0$.

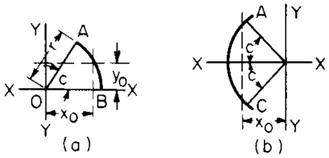


Fig. 3.1.17

Quadrant, *AB* (Fig. 3.1.18) $x_0 = y_0 = 2r/\pi = 0.6366r$.

Semicircumference, *AC* (Fig. 3.1.18) $y_0 = 2r/\pi = 0.6366r$; $x_0 = 0$.

Combination of Arcs and Straight Line (Fig. 3.1.19) *AD* and *BC* are two quadrants of radius *r*. $y_0 = \{(AB)r + 2[0.5\pi r(r - 0.6366r)]\} \div \{AB + 2(0.5\pi r)\}$.

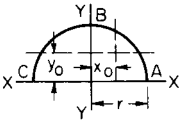


Fig. 3.1.18

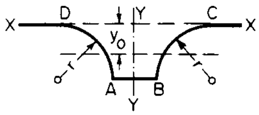


Fig. 3.1.19

CENTROIDS OF PLANE AREAS

Triangle Centroid lies at the intersection of the lines joining the vertices with the midpoints of the sides, and at a distance from any side equal to one-third of the corresponding altitude.

Parallelogram Centroid lies at the point of intersection of the diagonals.

Trapezoid (Fig. 3.1.20) Centroid lies on the line joining the middle points *m* and *n* of the parallel sides. The distances h_a and h_b are

$$h_a = h(a + 2b)/3(a + b) \qquad h_b = h(2a + b)/3(a + b)$$

Draw $BE = a$ and $CF = b$; *EF* will then intersect *mn* at centroid.

Any Quadrilateral The centroid of any quadrilateral may be determined by the general rule for areas, or graphically by dividing it into two sets of triangles by means of the diagonals. Find the centroid of each of the four triangles and connect the centroids of the triangles belonging to the same set. The intersection of these lines will be centroid of area. Thus, in Fig. 3.1.21, O, O_1, O_2, and O_3 are, respectively, the centroids of the triangles *ABD*, *ABC*, *BDC*, and *ACD*. The intersection of O_1O_3 with OO_2 gives the centroids.

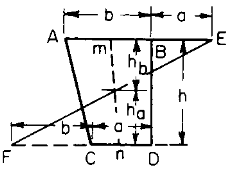


Fig. 3.1.20

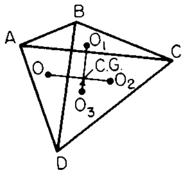


Fig. 3.1.21

Segment of a Circle (Fig. 3.1.22) $x_0 = \tfrac{2}{3}r \sin^3 c/(\text{rad } c - \cos c \sin c)$. A segment may be considered to be a sector from which a triangle is subtracted, and the general rule applied.

Sector of a Circle (Fig. 3.1.23) $x_0 = \tfrac{2}{3}r \sin c/\text{rad } c$; $y_0 = \tfrac{4}{3}r \sin^2 \tfrac{1}{2}c/\text{rad } c$.

Semicircle $x_0 = \tfrac{4}{3}r/\pi = 0.4244r$; $y_0 = 0$.

Quadrant (90° sector) $x_0 = y_0 = \tfrac{4}{3}r/\pi = 0.4244r$.

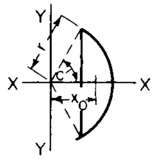


Fig. 3.1.22

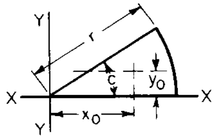


Fig. 3.1.23

Parabolic Half Segment (Fig. 3.1.24) Area *ABO*: $x_0 = \tfrac{3}{5}x_1$; $y_0 = \tfrac{3}{8}y_1$.

Parabolic Spandrel (Fig. 3.1.24) Area *AOC*: $x'_0 = \tfrac{3}{10}x_1$; $y'_0 = \tfrac{3}{4}y_1$.

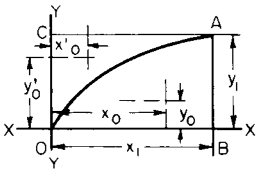


Fig. 3.1.24

Quadrant of an Ellipse (Fig. 3.1.25) Area *OAB*: $x_0 = \tfrac{4}{3}(a/\pi)$; $y_0 = \tfrac{4}{3}(b/\pi)$.

The centroid of a figure such as that shown in Fig. 3.1.26 may be determined as follows: Divide the area *OABC* into a number of parts by lines drawn perpendicular to the axis *XX*, e.g., 11, 22, 33, etc. These parts will be approximately either triangles, rectangles, or trapezoids. The area of each division may be obtained by taking the product of its mean height and its base. The centroid of each area may be obtained as previously shown. The sum of the moments of all the areas about *XX* and *YY*, respectively, divided by the sum of the areas will give approximately the distances from the center of gravity of the whole area to the axes *XX* and *YY*. The greater the number of areas taken, the more nearly exact the result.

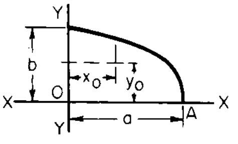


Fig. 3.1.25

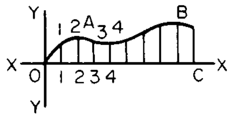


Fig. 3.1.26

Centroids of Solids

Prism or Cylinder with Parallel Bases The centroid lies in the center of the line connecting the centers of gravity of the bases.

Oblique Frustum of a Right Circular Cylinder (Fig. 3.1.27) Let 1 2 3 4 be the plane of symmetry. The distance from the base to the centroid is $\tfrac{1}{2}h + (r^2 \tan^2 c)/8h$, where c is the angle of inclination of the oblique section to the base. The distance of the centroid from the axis of the cylinder is $r^2 \tan c/4h$.

Pyramid or Cone The centroid lies in the line connecting the centroid of the base with the vertex and at a distance of one-fourth of the altitude above the base.

Truncated Pyramid If h is the height of the truncated pyramid and A and B the areas of its bases, the distance of its centroid from the surface of A is

$$h(A + 2\sqrt{AB} + 3B)/4(A + \sqrt{AB} + B)$$

Truncated Circular Cone If h is the height of the frustum and R and r the radii of the bases, the distance from the surface of the base whose radius is R to the centroid is $h(R^2 + 2Rr + 3r^2)/4(R^2 + Rr + r^2)$.

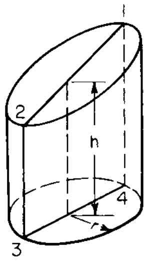


Fig. 3.1.27

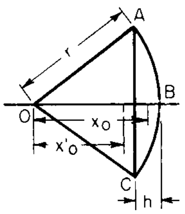


Fig. 3.1.28

Segment of a Sphere (Fig. 3.1.28) Volume *ABC*: $x_0 = 3(2r - h)^2/4(3r - h)$.

Hemisphere $x_0 = 3r/8$.

Hollow Hemisphere $x_0 = 3(R^4 - r^4)/8(R^3 - r^3)$, where R and r are, respectively, the outer and inner radii.

Sector of a Sphere (Fig. 3.1.28) Volume *OABCO*: $x'_0 = \tfrac{3}{8}(2r - h)$.

Ellipsoid, with Semiaxes a, b, and c For each octant, distance from center of gravity to each of the bounding planes = ⅜ × length of semiaxis perpendicular to the plane considered.

The formulas given for the determination of the centroid of lines and areas can be used to determine the areas and volumes of surfaces and solids of revolution, respectively, by employing the theorems of Pappus, Sec. 2.1.

Determination of Center of Gravity of a Body by Experiment The center of gravity may be determined by hanging the body up from different points and plumbing down; the point of intersection of the plumb lines will give the center of gravity. It may also be determined as shown in Fig. 3.1.29. The body is placed on knife-edges which rest on platform scales. The sum of the weights registered on the two scales $(w_1 + w_2)$ must equal the weight (w) of the body. Taking a moment axis at either end (say, O), $w_2A/w = x_0$ = distance from O to plane containing the center of gravity.

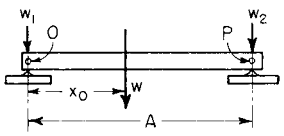


Fig. 3.1.29

Graphical Determination of the Centroids of Plane Areas See Fig. 3.1.40.

MOMENT OF INERTIA

The **moment of inertia of a solid body** with respect to a given axis is the limit of the sum of the products of the masses of each of the elementary particles into which the body may be conceived to be divided and the square of their distance from the given axis.

If $dm = dw/g$ represents the mass of an elementary particle and y its distance from an axis, the moment of inertia I of the body about this axis will be $I = \int y^2\, dm = \int y^2\, dw/g$.

The moment of inertia may be expressed in weight units ($I_w = \int y^2\, dw$), in which case the moment of inertia in weight units, I_w, is equal to the moment of inertia in mass units, I, multiplied by g.

If $I = k^2 m$, the quantity k is called the **radius of gyration** or the **radius of inertia.**

If a body is considered to be composed of a number of parts, its moment of inertia about an axis is equal to the sum of the moments of inertia of the several parts about the same axis, or $I = I_1 + I_2 + I_3 + \cdots + I_n$.

The **moment of inertia of an area** with respect to a given axis is the limit of the sum of the products of the elementary areas into which the area may be conceived to be divided and the square of their distance (y) from the axis in question. $I = \int y^2\, dA = k^2 A$, where k = **radius of gyration**.

The quantity $\int y^2\, dA$ is more properly referred to as the *second moment of area* since it is not a measure of *inertia* in a true sense.

Formulas for moments of inertia and radii of gyration of various areas follow later in this section.

Relation between the Moments of Inertia of an Area and a Solid The moment of inertia of a solid of elementary thickness about an axis is equal to the moment of inertia of the area of one face of the solid about the same axis multiplied by the mass per unit volume of the solid times the elementary thickness of the solid.

Moments of Inertia about Parallel Axes The moment of inertia of an area or solid about any given axis is equal to the moment of inertia about a parallel axis through the center of gravity plus the square of the distance between the two axes times the area or mass.

In Fig. 3.1.30*a*, the moment of inertia of the area *ABCD* about axis *YY* is equal to I_0 (or the moment of inertia about Y_0Y_0 through the center of gravity of the area and parallel to *YY*) plus x_0^2A, where A = area of *ABCD*. In Fig. 3.1.30*b*, the moment of inertia of the mass m about $YY = I_0 + x_0^2m$. Y_0Y_0 passes through the centroid of the mass and is parallel to *YY*.

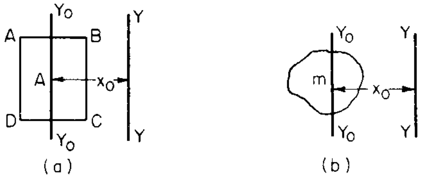


Fig. 3.1.30

Polar Moment of Inertia The polar moment of inertia (Fig. 3.1.31) is taken about an axis perpendicular to the plane of the area. Referring to Fig. 3.1.31, if I_y and I_x are the moments of inertia of the area A about *YY* and *XX*, respectively, then the polar moment of inertia $I_p = I_x + I_y$, or the polar moment of inertia is equal to the sum of the moments of inertia about any two axes at right angles to each other in the plane of the area and intersecting at the pole.

Product of Inertia This quantity will be represented by I_{xy}, and is $\int\int xy\, dy\, dx$, where x and y are the coordinates of any elementary part into which the area may be conceived to be divided. I_{xy} may be positive or negative, depending upon the position of the area with respect to the coordinate axes *XX* and *XY*.

Relation between Moments of Inertia about Axes Inclined to Each Other Referring to Fig. 3.1.32, let I_y and I_x be the moments of inertia of the area A about *YY* and *XX*, respectively, I'_y and I'_x the moments about $Y'Y'$ and $X'X'$, and I_{xy} and I'_{xy} the products of inertia for *XX* and *YY*, and $X'X'$ and $Y'Y'$, respectively. Also, let c be the angle between the respective pairs of axes, as shown. Then,

$$I'_y = I_y \cos^2 c + I_x \sin^2 c + I_{xy} \sin 2c$$
$$I'_x = I_x \cos^2 c + I_y \sin^2 c - I_{xy} \sin 2c$$
$$I'_{xy} = \frac{I_x - I_y}{2} \sin 2c + I_{xy} \cos 2c$$

Principal Moments of Inertia In every plane area, a given point being taken as the origin, there is at least one pair of rectangular axes in the plane of the area about one of which the moment of inertia is a maximum, and a minimum about the other. These moments of inertia are called the **principal moments of inertia,** and the axes about which they are taken are the **principal axes of inertia.** One of the conditions for principal moments of inertia is that the product of inertia I_{xy} shall equal zero. **Axes of symmetry** of an area are always principal axes of inertia.

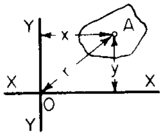


Fig. 3.1.31

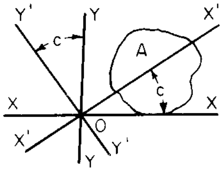


Fig. 3.1.32

Relation between Products of Inertia and Parallel Axes In Fig. 3.1.33, X_0X_0 and Y_0Y_0 pass through the center of gravity of the area parallel to the given axes *XX* and *YY*. If I_{xy} is the product of inertia for *XX* and *YY*, and $I_{x_0y_0}$ that for X_0X_0 and Y_0Y_0, then $I_{xy} = I_{x_0y_0} + abA$.

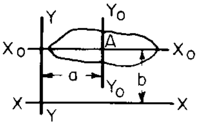


Fig. 3.1.33

Mohr's Circle The **principal moments of inertia** and the location of the **principal axes of inertia** for any point of a plane area may be established graphically as follows.

Given at any point A of a plane area (Fig. 3.1.34), the moments of inertia I_x and I_y about axes X and Y, and the product of inertia I_{xy} relative to X and Y. The graph shown in Fig. 3.1.34*b* is plotted on rectangular coordinates with moments of inertia as abscissas and products of inertia as ordinates. Lay out $Oa = I_x$ and $ab = I_{xy}$ (upward for positive products of inertia, downward for negative). Lay out $Oc = I_y$ and cd = *negative of* I_{xy}. Draw a circle with *bd* as diameter. This is **Mohr's circle**. The *maximum* moment of inertia is $I'_x = Of$; the *minimum* moment of inertia is $I'_y = Og$. The principal axes of inertia are located as follows. From axis *AX* (Fig. 3.1.34*a*) lay out angular distance $\theta = ½ < bef$. This locates axis AX', one principal axis ($I'_x = Of$). The other principal axis of inertia is AY', perpendicular to AX' ($I'_x = Og$).

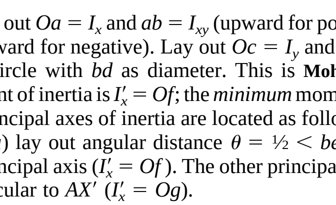


Fig. 3.1.34

The **moment of inertia of any area** may be considered to be made up of the sum or difference of the known moments of inertia of simple figures. For example, the dimensioned figure shown in Fig. 3.1.35 represents the section of a rolled shape with hole *oprs* and may be divided into the semicircle *abc*, rectangle *edkg*, and triangles *mfg* and *hkl*, from which the rectangle *oprs* is to be subtracted. Referring to axis *XX*,

$$\begin{aligned} I_{xx} &= \pi 4^4/8 \text{ for semicircle } abc = (2 \times 11^3)/3 \text{ for rectangle } edkg \\ &= 2[(5 \times 3^3)/36 + 10^2(5 \times 3)/2] \text{ for the two triangles } mfg \\ &\quad \text{and } hkl \end{aligned}$$

From the sum of these there is to be subtracted $I_{xx} = [(2 \times 3^2)/12 + 4^2(2 \times 3)]$ for the rectangle *oprs*.

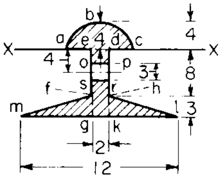


Fig. 3.1.35

If the moment of inertia of the whole area is required about an axis parallel to *XX*, but passing through the center of gravity of the whole area, $I_0 = I_{xx} - x_0^2 A$, where x_0 = distance from *XX* to center of gravity. The **moments of inertia of built-up sections** used in structural work may be found in the same manner, the moments of inertia of the different **rolled sections** being given in Sec. 12.2.

Moments of Inertia of Solids For moments of inertia of solids about parallel axes, $I_x = I_0 + x_0^2 m$.

Moment of Inertia with Reference to Any Axis Let a mass particle *dm* of a body have *x*, *y*, and *z* as coordinates, *XX*, *YY*, and *ZZ* being the coordinate axes and *O* the origin. Let *X′X′* be any axis passing through the origin and making angles of *A*, *B*, and *C* with *XX*, *YY*, and *ZZ*, respectively. The moment of inertia with respect to this axis then becomes equal to

$$\begin{aligned} I_x' = \cos^2 A \textstyle\int (y^2 + z^2)\, dm + \cos^2 B \int (z^2 + x^2)\, dm \\ + \cos^2 C \textstyle\int (x^2 + y^2)\, dm - 2 \cos B \cos C \int yz\, dm \\ - 2 \cos C \cos A \textstyle\int zx\, dm - 2 \cos A \cos B \int xy\, dm \end{aligned}$$

Let the moment of inertia about $XX = I_x = \int (y^2 + z^2)\, dm$, about $YY = I_y = \int (z^2 + x^2)\, dm$, and about $ZZ = I_z = \int (x^2 + y^2)\, dm$. Let the products of inertia about the three coordinate axes be

$$I_{yz} = \int yz\, dm \qquad I_{zx} = \int zx\, dm \qquad I_{xy} = \int xy\, dm$$

Then the moment of inertia I_x' becomes equal to

$$\begin{aligned} I_x \cos^2 A + I_y \cos^2 B + I_z \cos^2 C - 2I_{yz} \cos B \cos C - 2I_{zx} \\ \cos C \cos A - 2I_{xy} \cos A \cos B \end{aligned}$$

The moment of inertia of any solid may be considered to be made up of the sum or difference of the moments of inertia of simple solids of which the moments of inertia are known.

Moments of Inertia of Important Solids (Homogeneous)

$m = w/g$ = mass per unit of volume of the body
$M = W/g$ = total mass of body
r = radius
I = moment of inertia (mass units)
$I_w = I \times g$ = moment of inertia (weight units)

Solid circular cylinder about its axis: $I = \pi r^4 ma/2 = Mr^2/2$. ($a$ = length of axis of cylinder.)

Solid circular cylinder about an axis through the center of gravity and perpendicular to axis of cylinder: $I = M[r^2 + (a^2/3)]/4$.

Hollow circular cylinder about its axis: $I = \pi ma(r_1^4 - r_2^4)/2$. ($r_1$ and r_2 = outer and inner radii; a = length.)

Thin hollow circular cylinder about its axis: $I = Mr^2$.

Solid sphere about a diameter: $I = 8m\pi r^5/15 = 2Mr^2/5$.

Thin hollow sphere about a diameter: $I = 2Mr^2/3$.

Thick hollow sphere about a diameter: $I = 8m\pi(r_1^5 - r_2^5)/15$. ($r_1$ and r_2 are outer and inner radii.)

Rectangular prism about an axis through center of gravity and perpendicular to a face whose dimensions are a and b: $I = M(a^2 + b^2)/12$.

Solid right circular cone about an axis through its apex and perpendicular to its axis: $I = 3M[(r^2/4) + h^2]/5$. (h = altitude of cone, r = radius of base.)

Solid right circular cone about its axis of revolution: $I = 3Mr^2/10$.

Ellipsoid with semiaxes a, b, and c: I about diameter $2c$ (z axis) = $4m\pi abc\ (a^2 + b^2)/15$. [Equation of ellipsoid: $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.]

Ring with Circular Section (Fig. 3.1.36) $I_{yy} = ½m\pi^2 Ra^2(4R^2 + 3a^2)$; $I_{xx} = m\pi^2 Ra^2[R^2 + (5a^2/4)]$.

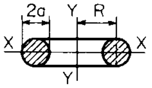


Fig. 3.1.36

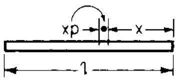


Fig. 3.1.37

Approximate Moments of Inertia of Solids In order to determine the moment of inertia of a solid, it is necessary to know all its dimensions. In the case of a rod of mass M (Fig. 3.1.37) and length l, with shape and size of the cross section unknown, making the approximation that the weight is all concentrated along the axis of the rod, the moment of inertia about *YY* will be $I_{yy} = \int_0^l (M/l)x^2\, dx = Ml^2/3$.

A **thin plate** may be treated in the same way (Fig. 3.1.38): $I_{yy} = \int_0^l (M/l)x^2\, dx$. Here the mass of the plate is assumed concentrated at its middle layer.

Thin Ring, or Cylinder (Fig. 3.1.39) Assume the mass M of the ring or cylinder to be concentrated at a distance r from O. The moment of inertia about an axis through O perpendicular to plane of ring or along the axis of the cylinder will be $I = Mr^2$; this will be greater than the exact moment of inertia, and r is sometimes taken as the distance from O to the center of gravity of the cross section of the rim.

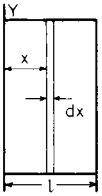


Fig. 3.1.38

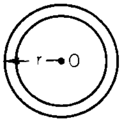


Fig. 3.1.39

Flywheel Effect The moment of inertia of a solid is often called flywheel effect in the solution of problems dealing with rotating bodies, and is usually expressed in lb · ft² (I_w).

Graphical Determination of the Centroids and Moments of Inertia of Plane Areas Required to find the center of gravity of the area *MNP* (Fig. 3.1.40) and its moment of inertia about any axis *XX*.

Draw any line *SS* parallel to *XX* and at a distance *d* from it. Draw a number of lines such as *AB* and *EF* across the figure parallel to *XX*. From *E* and *F* draw *ER* and *FT* perpendicular to *SS*. Select as a pole any

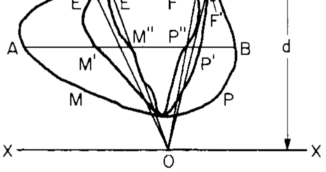


Fig. 3.1.40

point on XX, preferably the point nearest the area, and draw OR and OT, cutting EF at E' and F'. If the same construction is repeated, using other lines parallel to XX, a number of points will be obtained, which, if connected by a smooth curve, will give the area $M'N'P'$. Project E' and F' onto SS by lines $E'R'$ and $F'T'$. Join F' and T' with O, obtaining E'' and F''; connect the points obtained using other lines parallel to XX and obtain an area $M''N''P''$. The area $M'N'P' \times d =$ moment of area MNP about the line XX, and the distance from XX to the centroid $MNP =$ area $M'N'P' \times d/$area MNP. Also, area $M''N''P'' \times d^2 =$ moment of inertia of MNP about XX. The areas $M'N'P'$ and $M''N''P''$ can best be obtained by use of a planimeter.

KINEMATICS

Kinematics is the study of the motion of bodies without reference to the forces causing that motion or the mass of the bodies.

The **displacement** of a point is the directed distance that a point has moved on a geometric path from a convenient origin. It is a **vector,** having both magnitude and direction, and is subject to all the laws and characteristics attributed to vectors. In Fig. 3.1.41, the displacement of the point A from the origin O is the directed distance O to A, symbolized by the vector **s**.

The **velocity** of a point is the time rate of change of displacement, or $\mathbf{v} = d\mathbf{s}/dt$.

The **acceleration** of a point is the time rate of change of velocity, or $\mathbf{a} = d\mathbf{v}/dt$.

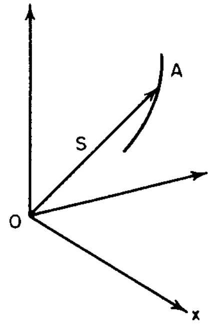


Fig. 3.1.41

The kinematic definitions of velocity and acceleration involve the four variables, *displacement, velocity, acceleration,* and *time.* If we eliminate the variable of time, a third equation of motion is obtained, $ds/v = dt = dv/a$. This differential equation, together with the definitions of velocity and acceleration, make up the *three kinematic equations* of motion, $v = ds/dt$, $a = dv/dt$, and $a\ ds = v\ dv$. These differential equations are usually limited to the scalar form when expressed together, since the last can only be properly expressed in terms of the scalar dt. The first two, since they are definitions for velocity and acceleration, are **vector equations.**

A **space-time curve** offers a convenient means for the study of the motion of a point. The **slope of the curve** at any point will represent the **velocity** at that time. In Fig. 3.1.42*a* the slope is constant, as the graph is a straight line; the velocity is therefore uniform. In Fig. 3.1.42*b* the slope of the curve varies from point to point, and the velocity must also vary. At p and q the slope is zero; therefore, the velocity of the point at the corresponding times must also be zero.

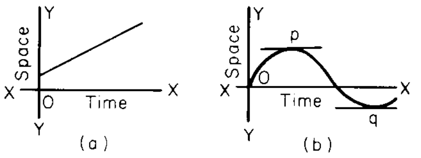


Fig. 3.1.42

A **velocity-time curve** offers a convenient means for the study of acceleration. The **slope of the curve** at any point will represent the **acceleration** at that time. In Fig. 3.1.43*a* the slope is constant; so the acceleration must be constant. In the case represented by the full line, the acceleration is positive; so the velocity is increasing. The dotted line shows a negative acceleration and therefore a decreasing velocity. In Fig. 3.1.43*b* the slope of the curve varies from point to point; so the acceleration must also vary. At p and q the slope is zero; therefore, the acceleration of the point at the corresponding times must also be zero. The area under the velocity-time curve between any two ordinates such as NL and HT will represent the distance moved in time interval LT. In the case of the uniformly accelerated motion shown by the full line in Fig. 3.1.43*a*, the area $LNHT$ is $\frac{1}{2}(NL + HT) \times (OT - OL) =$ mean velocity multiplied by the time interval = space passed over during this time interval. In Fig. 3.1.43*b* the mean velocity can be obtained from the equation of the curve by means of the calculus, or graphically by approximation of the area.

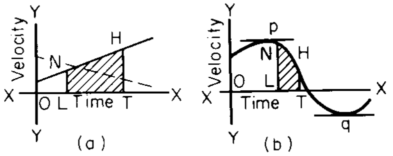


Fig. 3.1.43

An **acceleration-time curve** (Fig. 3.1.44) may be constructed by plotting accelerations as ordinates, and times as abscissas. The area under this curve between any two ordinates will represent the total increase in velocity during the time interval. The area $ABCD$ represents the total increase in velocity between time t_1 and time t_2.

General Expressions Showing the Relations between Space, Time, Velocity, and Acceleration for Rectilinear Motion

SPECIAL MOTIONS

Uniform Motion If the **velocity** is **constant,** the **acceleration** must be **zero,** and the point has uniform motion. The space-time curve becomes a straight line inclined toward the time axis (Fig. 3.1.42*a*). The velocity-time curve becomes a straight line parallel to the time axis. For this motion $a = 0$, $v =$ constant, and $s = s_0 + vt$.

Uniformly Accelerated or Retarded Motion If the **velocity** is **not uniform** but the **acceleration** is **constant,** the point has uniformly accelerated motion; the acceleration may be either positive or negative. The space-time curve becomes a parabola and the velocity-time curve becomes a straight line inclined toward the time axis (Fig. 3.1.43*a*). The acceleration-time curve becomes a straight line parallel to the time axis. For this motion $a =$ constant, $v = v_0 + at$, $s = s_0 + v_0t + \frac{1}{2}at^2$.

If the point starts from rest, $v_0 = 0$. Care should be taken concerning the sign + or − for acceleration.

Composition and Resolution of Velocities and Acceleration

Resultant Velocity A velocity is said to be the resultant of two other velocities when it is represented by a vector that is the geometric sum of the vectors representing the other two velocities. This is the **parallelogram of motion.** In Fig. 3.1.45, **v** is the resultant of $\mathbf{v}_1$ and $\mathbf{v}_2$ and is

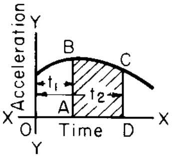


Fig. 3.1.44

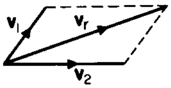


Fig. 3.1.45

represented by the diagonal of a parallelogram of which $\mathbf{v}_1$ and $\mathbf{v}_2$ are the sides; or it is the third side of a triangle of which $\mathbf{v}_1$ and $\mathbf{v}_2$ are the other two sides.

Polygon of Motion The parallelogram of motion may be extended to the polygon of motion. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ (Fig. 3.1.46*a*) show the directions of four velocities imparted in the same plane to point *O*. If the lines $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ (Fig. 3.1.46*b*) are drawn parallel to and proportional to the velocities imparted to point *O*, $\mathbf{v}$ will represent the resultant velocity imparted to *O*. It will make no difference in what order the velocities are taken in constructing the motion polygon. As long as the arrows showing the direction of the motion follow each other in order about the polygon, the resultant velocity of the point will be represented in magnitude by the closing side of the polygon, but opposite in direction.

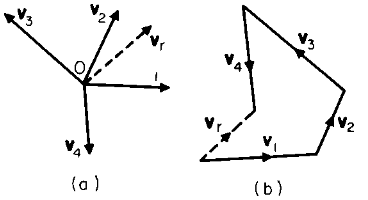


Fig. 3.1.46

Resolution of Velocities Velocities may be resolved into **component velocities** in the same plane, as shown by Fig. 3.1.47. Let the velocity of point *O* be v_r. In Fig. 3.1.47*a* this velocity is resolved into two components in the same plane as v_r and at right angles to each other.

$$v_r = \sqrt{(v_1)^2 + (v_2)^2}$$

In Fig. 3.1.47*b* the components are in the same plane as v_r, but are not at right angles to each other. In this case,

$$v_r = \sqrt{(v_1)^2 + (v_2)^2 + 2v_1v_2 \cos B}$$

If the components v_1 and v_2 and angle *B* are known, the direction of v_r can be determined. $\sin bOc = (v_1/v_r) \sin B$. $\sin cOa = (v_2/v_r) \sin B$. Where v_1 and v_2 are at right angles to each other, $\sin B = 1$.

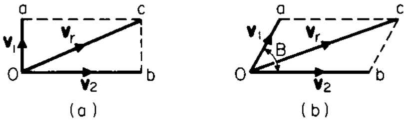


Fig. 3.1.47

Resultant Acceleration Accelerations may be combined and resolved in the same manner as velocities, but in this case the lines or vectors represent accelerations instead of velocities. If the acceleration had components of magnitude a_1 and a_2, the magnitude of the resultant acceleration would be $a = \sqrt{(a_1)^2 + (a_2)^2 + 2a_1a_2 \cos B}$, where *B* is the angle between the vectors a_1 and a_2.

Curvilinear Motion in a Plane

The linear velocity $v = ds/dt$ of a point in curvilinear motion is the same as for rectilinear motion. Its direction is tangent to the path of the point. In Fig. 3.1.48*a*, let $P_1P_2P_3$ be the path of a moving point and $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ represent its velocity at points P_1, P_2, P_3, respectively. If *O* is taken as a pole (Fig. 3.1.48*b*) and vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ representing the velocities of the point at P_1, P_2, and P_3 are drawn, the curve connecting the terminal points of these vectors is known as the **hodograph** of the motion. This velocity diagram is applicable only to motions all in the same plane.

Acceleration Tangents to the curve (Fig. 3.1.48*b*) indicate the directions of the **instantaneous velocities**. The direction of the tangents does not, as a rule, coincide with the direction of the accelerations as represented by tangents to the path. If the acceleration *a* at some point in the path is resolved by means of a parallelogram into components tangent and normal to the path, the normal acceleration $a_n = v^2/\rho$, where ρ = radius of curvature of the path at the point in question, and the tangential acceleration $a_t = dv/dt$, where v = velocity tangent to the path at the same point. $a = \sqrt{a_n^2 + a_t^2}$. The normal acceleration is constantly directed toward the center of the path.

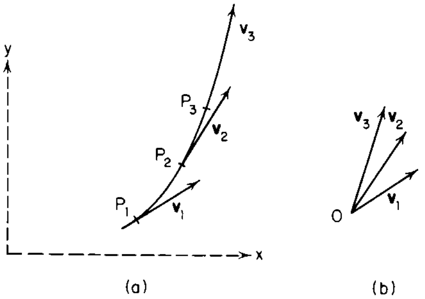


Fig. 3.1.48

EXAMPLE. Figure 3.1.49 shows a point moving in a curvilinear path. At p_1 the velocity is $\mathbf{v}_1$; at p_2 the velocity is $\mathbf{v}_2$. If these velocities are drawn from pole *O* (Fig. 3.1.49*b*), $\Delta\mathbf{v}$ will be the difference between $\mathbf{v}_2$ and $\mathbf{v}_1$. The acceleration during travel p_1p_2 will be $\Delta\mathbf{v}/\Delta t$, where Δt is the time interval. The approximation becomes closer to instantaneous acceleration as shorter intervals Δt are employed.

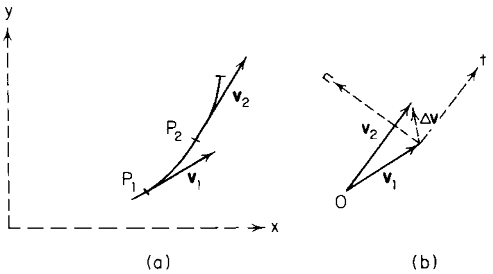


Fig. 3.1.49

The acceleration $\Delta\mathbf{v}/\Delta t$ can be resolved into normal and tangential components leading to $\mathbf{a}_n = \Delta\mathbf{v}_n/\Delta t$, normal to the path, and $\mathbf{a}_r = \Delta\mathbf{v}_p/\Delta t$, tangential to the path.

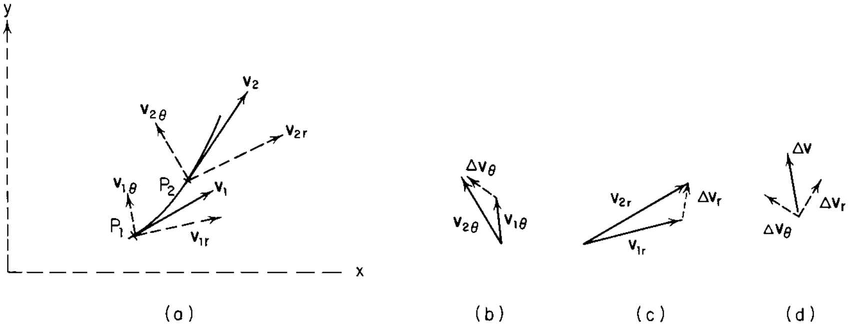
Velocity and *acceleration* may be expressed in **polar coordinates** such that $v = \sqrt{v_r^2 + v_\theta^2}$ and $a = \sqrt{a_r^2 + a_\theta^2}$. Figure 3.1.50 may be used to explain the *r* and θ coordinates.

EXAMPLE. At P_1 the velocity is $\mathbf{v}_1$, with components $\mathbf{v}_{1r}$ in the *r* direction and $\mathbf{v}_{1\theta}$ in the θ direction. At P_2 the velocity is $\mathbf{v}_2$, with components $\mathbf{v}_{2r}$ in the *r* direction and $\mathbf{v}_{2\theta}$ in the θ direction. It is evident that the difference in velocities $\mathbf{v}_2 - \mathbf{v}_1 = \Delta\mathbf{v}$ will have components $\Delta\mathbf{v}_r$ and $\Delta\mathbf{v}_\theta$, giving rise to accelerations $\mathbf{a}_r$ and $\mathbf{a}_\theta$ in a time interval Δt.

In polar coordinates, $v_\mathbf{r} = dr/dt$, $a_\mathbf{r} = d^2r/dt^2 - r(d\theta/dt)^2$, $v_\theta = r(d\theta/dt)$, and $a_\theta = r(d^2\theta/dt^2) + 2(dr/dt)(d\theta/dt)$.

If a point *P* moves on a circular path of radius *r* with an angular velocity of ω and an angular acceleration of α, the linear velocity of the point *P* is $v = \omega r$ and the two components of the linear acceleration are $a_n = v^2/r = \omega^2 r = v\omega$ and $a_t = \alpha r$.

If the angular velocity is constant, the point *P* travels equal circular paths in equal intervals of time. The projected displacement, velocity, and acceleration of the point *P* on the *x* and *y* axes are sinusoidal functions of time, and the motion is said to be **harmonic motion**. Angular velocity is usually expressed in radians per second, and when the number (*N*) of revolutions traversed per minute (r/min) by the point *P* is known, the angular velocity of the radius *r* is $\omega = 2\pi N/60 = 0.10472N$.



(a) (b) (c) (d)

Fig. 3.1.50

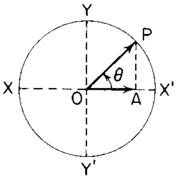
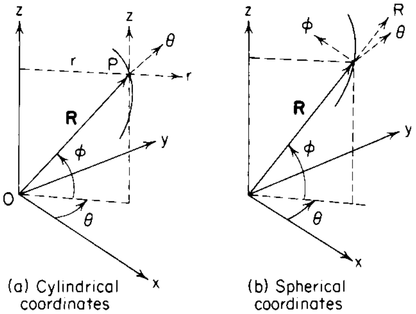


Fig. 3.1.51

In Fig. 3.1.51, let the angular velocity of the line OP be a constant ω. Let the point P start at X' and move to P in time t. Then the angle $\theta = \omega t$. If $OP = r$, $X'A = r - OA = r - r\cos\omega t = s$. The velocity V of the point A on the x axis will equal $ds/dt = \omega r \sin \omega t$, and the acceleration $a = dv/dt = -\omega^2 r \cos \omega t$. The period τ is the time necessary for the point P to complete one cycle of motion $\tau = 2\pi/\omega$, and it is also equal to the time necessary for A to complete a full cycle on the x axis from X' to X and return.

Curvilinear Motion in Space

If **three dimensions** are used, velocities and accelerations may be resolved into components not in the same plane by what is known as the **parallelepiped of motion.** Three coordinate systems are widely used, cartesian, cylindrical, and spherical. In **cartesian coordinates,** $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ and $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$. In **cylindrical coordinates,** the radius vector **R** of displacement lies in the rz plane, which is at an angle with the xz plane. Referring to (a) of Fig. 3.1.52, the θ coordinate is perpendicular to the rz plane. In this system $v = \sqrt{v_r^2 + v_\theta^2 + v_z^2}$ and $a = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$ where $\mathrm{v}_r = dr/dt$, $\mathrm{a}_r = d^2r/dt^2 - r(d\theta/dt)^2$, $v_\theta = r(d\theta/dt)$, and $a_\theta = r(d^2\theta/dt^2) + 2(dr/dt)(d\theta/dt)$. In **spherical coordinates,** the three coordinates are the R coordinate, the θ coordinate, and the ϕ coordinate as in (b) of Fig. 3.1.52. The velocity and acceleration are $v = \sqrt{v_R^2 + v_\theta^2 + v_\phi^2}$ and $a = \sqrt{a_R^2 + a_\theta^2 + a_\phi^2}$, where $v_R = dR/dt$, $v_\phi = R(d\phi/dt)$, $v_\theta = R\cos\phi(d\theta/dt)$, $a_R = d^2R/dt^2 - R(d\phi/dt)^2 - R\cos^2\phi(d\theta/dt)^2$, $a_\phi = R(d^2\phi/dt^2) + R\cos\phi\sin\phi\,(d\theta/dt)^2 + 2(dR/dt)(d\phi/dt)$, and $a_\theta = R\cos\phi\,(d^2\theta/dt^2) + 2[(dR/dt)\cos\phi - R\sin\phi\,(d\phi/dt)]\,d\theta/dt$.



(a) Cylindrical coordinates

(b) Spherical coordinates

Fig. 3.1.52

Motion of Rigid Bodies

A body is said to be **rigid** when the distances between all its particles are invariable. Theoretically, rigid bodies do not exist, but materials used in engineering are rigid under most practical working conditions. The motion of a rigid body can be completely described by knowing the **angular motion** of a line on the rigid body and the **linear motion** of a point on this line and relating the motion of all other parts of the rigid body to these motions. If a rigid body moves so that a straight line connecting any two of its particles remains parallel to its original position at all times, it is said to have **translation.** In **rectilinear translation,** all points move in straight lines. In **curvilinear translation,** all points move on congruent curves but without rotation. **Rotation** is defined as angular motion about an axis, which may or may not be fixed. Rigid body motion in which the paths of all particles lie on parallel planes is called **plane motion.**

Angular Motion

Angular displacement is the change in angular position of a given line as measured from a convenient reference line. In Fig. 3.1.53, consider the motion of the line AB as it moves from its original position $A'B'$. The angle between lines AB and $A'B'$ is the angular displacement of line **AB,** symbolized as θ. It is a directed quantity and is a vector. The usual notation used to designate angular displacement is a vector normal to the plane in which the angular displacement occurs. The length of the vector is proportional to the magnitude of the angular displacement. For a rigid body moving in three dimensions, the line AB may have angular motion about any three orthogonal axes. For example, the angular displacement can be described in cartesian coordinates as $\theta = \theta_x + \theta_y + \theta_z$, where $\theta = \sqrt{\theta_x^2 + \theta_y^2 + \theta_z^2}$.

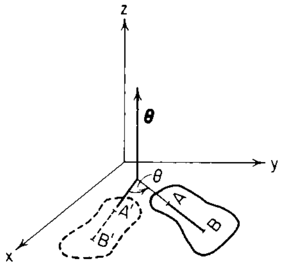


Fig. 3.1.53

Angular velocity is defined as the time rate of change of angular displacement, $\omega = d\theta/dt$. Angular velocity may also have components about any three orthogonal axes.

Angular acceleration is defined as the time rate of change of angular velocity, $\alpha = d\omega/dt = d^2\theta dt^2$. Angular acceleration may also have components about any three orthogonal axes.

The *kinematic equations* of angular motion of a line are analogous to those for the motion of a point. In referring to Table 3.1.1, $\omega = d\theta/dt$ $\alpha = d\omega/dt$, and $\alpha\, d\theta = \omega\, d\omega$. Substitute θ for s, ω for v, and α for a.

Motion of a Rigid Body in a Plane

Plane motion is the motion of a rigid body such that the paths of all particles of that rigid body lie on parallel planes.

Table 3.1.1

Variables	$s = f(t)$	$v = f(t)$	$a = f(t)$	$a = f(s,v)$
Displacement		$s = s_0 + \int_{t_0}^{t} v\,dt$	$s = s_0 + \int_{t_0}^{t}\int_{t_0}^{t} a\,dt\,dt$	$s = s_0 + \int_{v_0}^{v} (v/a)\,dv$
Velocity	$v = ds/dt$		$v = v_0 + \int_{t_0}^{t} a\,dt$	$\int_{v_0}^{v} v\,dv = \int_{s}^{s_0} a\,ds$
Acceleration	$a = d^2s/dt^2$	$a = dv/dt$		$a = v\,dv/ds$

Instantaneous Axis When the axis about which any body may be considered to rotate changes its position, any one position is known as an instantaneous axis, and the line through all positions of the instantaneous axis as the **centrode.**

When the velocity of two points in the same plane of a rigid body having plane motion is known, the instantaneous axis for the body will be at the intersection of the lines drawn from each point and perpendicular to its velocity. See Fig. 3.1.54, in which *A* and *B* are two points on the rod *AB*, $\mathbf{v}_1$ and $\mathbf{v}_2$ representing their velocities. *O* is the instantaneous axis for *AB*; therefore point *C* will have velocity shown in a line perpendicular to *OC*.

Linear velocities of points in a body rotating about an instantaneous axis are proportional to their distances from this axis. In Fig. 3.1.54, $\mathbf{v}_1 : \mathbf{v}_2 : \mathbf{v}_3 = AO : OB : OC$. If the velocities of *A* and *B* were parallel, the lines *OA* and *OB* would also be parallel and there would be no instantaneous axis. The motion of the rod would be translation, and all points would be moving with the same velocity in parallel straight lines.

If a body has plane motion, the components of the velocities of any two points in the body along the straight line joining them must be equal. *Ax* must be equal to *By* and *Cz* in Fig. 3.1.54.

EXAMPLE. In Fig. 3.1.55*a*, the velocities of points *A* and *B* are known—they are $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. To find the instantaneous axis of the body, perpendiculars *AO* and *BO* are drawn. *O*, at the intersection of the perpendiculars, is the **instantaneous axis** of the body. To find the velocity of any other point, like *C*, line *OC* is drawn and v_3 erected perpendicular to *OC* with magnitude equal to v_1 (*CO/AO*). The **angular velocity** of the body will be $\omega = v_1/AO$ or v_2/BO or v_3/CO. The instantaneous axis of a wheel rolling on a rack without slipping (Fig. 3.1.55*b*) lies at the point of contact *O*, which has zero linear velocity. All points of the wheel will have velocities perpendicular to radii to *O* and proportional in magnitudes to their respective distances from *O*.

Another way to describe the plane motion of a rigid body is with the use of **relative motion.** In Fig. 3.1.56 the velocity of point *A* is $\mathbf{v}_1$. The angular velocity of the line *AB* is v_1/r_{AB}. The velocity of *B* relative to *A* is $\omega_{AB} \times r_{AB}$. Point *B* is considered to be moving on a circular path around *A* as a center. The direction of relative velocity of *B* to *A* would be tangent to the circular path in the direction that ω_{AB} would make *B* move. The velocity of *B* is the vector sum of the velocity *A* added to the velocity of *B* relative to *A*, $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$.

The **acceleration** of *B* is the vector sum of the acceleration of *A* added to the acceleration of *B* relative to *A*, $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$. Care must be taken 0to include the complete relative acceleration of *B* to *A*. If *B* is considered to move on a circular path about *A*, with a velocity relative to *A*, it will have an acceleration relative to *A* that has both normal and tangential components: $\mathbf{a}_{B/A} = (\mathbf{a}_{B/A})_n + (\mathbf{a}_{B/A})_t$.

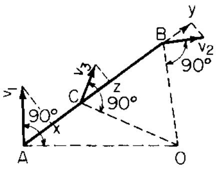
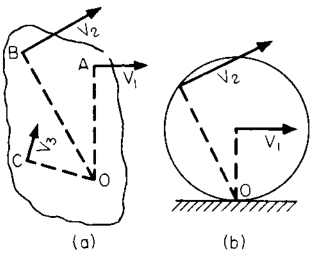
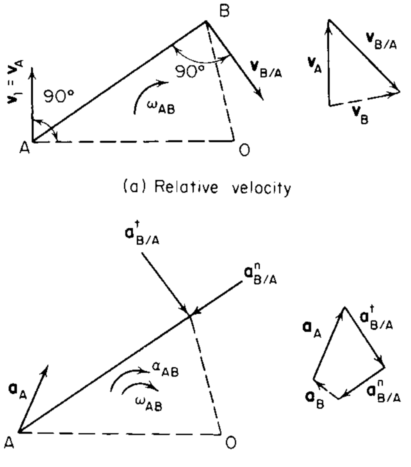


Fig. 3.1.54



(a) (b)

Fig. 3.1.55



(a) Relative velocity

(b) Relative acceleration

Fig. 3.1.56

If *B* is a point on a path which lies on the same rigid body as the line *AB*, a **particle** *P* **traveling on the path** will have a velocity $\mathbf{v}_P$ at the instant *P* passes over point *B* such that $\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{B/A} + \mathbf{v}_{P/B}$, where the velocity $\mathbf{v}_{P/B}$ is the velocity of *P* relative to path *B*.

The particle *P* will have an acceleration $\mathbf{a}_P$ at the instant *P* passes over the point *B* such that $\mathbf{a}_P = \mathbf{a}_A + \mathbf{a}_{B/A} + \mathbf{a}_{P/B} + 2\omega_{AB} \times \mathbf{v}_{P/B}$. The term $\mathbf{a}_{P/B}$ is the acceleration of *P* relative to the path at point *B*. The last term $2\omega_{AB}\,\mathbf{v}_{P/B}$ is frequently referred to as the **coriolis acceleration.** The direction is always normal to the path in a sense which would rotate the head of the vector $\mathbf{v}_{P/B}$ about its tail in the direction of the angular velocity of the rigid body ω_{AB}.

EXAMPLE. In Fig. 3.1.57, arm *AB* is rotating counterclockwise about *A* with a constant angular velocity of 38 r/min or 4 rad/s, and the slider moves outward with a velocity of 10 ft/s (3.05 m/s). At an instant when the slider *P* is 30 in (0.76 m) from the center *A*, the acceleration of the slider will have two components. One component is the normal acceleration directed toward the center *A*. Its magnitude is $\omega^2 r = 4^2\ (30/12) = 40$ ft/s² [$\omega^2 r = 4^2\ (0.76) = 12.2$ m/s²]. The second is the coriolis acceleration directed normal to the arm *AB*, upward and to the left. Its magnitude is $2\omega v = 2(4)(10) = 80$ ft/s² [$2\omega v = 2(4)(3.05) = 24.4$ m/s²].

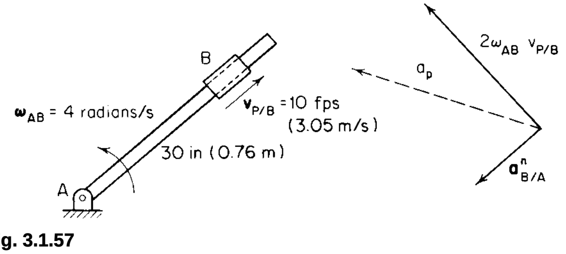


Fig. 3.1.57

General Motion of a Rigid Body

The general motion of a point moving in a coordinate system which is itself in motion is complicated and can best be summarized by using vector notation. Referring to Fig. 3.1.58, let the point P be displaced a vector distance $\mathbf{R}$ from the origin O of a moving reference frame x, y, z which has a velocity $\mathbf{v}_o$ and an acceleration $\mathbf{a}_o$. If point P has a velocity and an acceleration relative to the moving reference plane, let these be $\mathbf{v}_r$ and $\mathbf{a}_r$. The angular velocity of the moving reference fame is ω, and the origin of the moving reference frame is displaced a vector distance $\mathbf{R}_1$ from the origin of a primary (fixed) reference frame X, Y, Z. The velocity and acceleration of P are $\mathbf{v}_P = \mathbf{v}_o + \omega \times \mathbf{R} + \mathbf{v}_r$ and $\mathbf{a}_P = \mathbf{a}_o + (d\omega/dt) \times \mathbf{R} + \omega \times (\omega \times \mathbf{R}) + 2\omega \times \mathbf{v}_r + \mathbf{a}_r$.

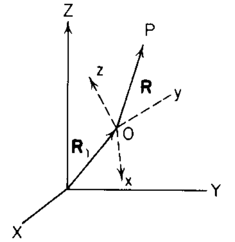


Fig. 3.1.58

DYNAMICS OF PARTICLES

Consider a particle of mass m subjected to the action of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , whose vector resultant is $\mathbf{R} = \Sigma\mathbf{F}$. According to Newton's first law of motion, if $\mathbf{R} = 0$, the body is acted on by a balanced force system, and it will either remain at rest or move uniformly in a straight line. If $\mathbf{R} \neq 0$, Newton's second law of motion states that the body will accelerate in the direction of and proportional to the magnitude of the resultant R. This may be expressed as $\Sigma\mathbf{F} = m\mathbf{a}$. If the resultant of the force system has components in the x, y, and z directions, the resultant acceleration will have proportional components in the x, y, and z direction so that $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$. If the resultant of the force system varies with time, the acceleration will also vary with time.

In **rectilinear motion,** the acceleration and the direction of the unbalanced force must be in the direction of motion. **Forces must be in balance and the acceleration equal to zero in any direction other than the direction of motion.**

EXAMPLE 1. The body in Fig. 3.1.59 has a mass of 90 lbm (40.8 kg) and is subjected to an external horizontal force of 36 lbf (160 N) applied in the direction shown. The coefficient of friction between the body and the inclined plane is 0.1. Required, the velocity of the body at the end of 5 s, if it starts from rest.

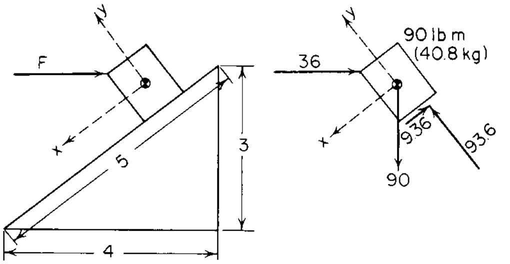


Fig. 3.1.59

First determine all the forces acting externally on the body. These are the applied force $F = 36$ lbf (106 N), the weight $W = 90$ lbf (400 N), and the force with which the plane reacts on the body. The latter force can be resolved into component forces, one normal and one parallel to the surface of the plane. Motion will be downward along the plane since a static analysis will show that the body will slide downward unless the static coefficient of friction is greater than 0.269. In the direction normal to the surface of the plane, the forces must be balanced. The normal force is $(3/5)(36) + (4/5)(90) = 93.6$ lbf (416 N). The frictional force is $93.6 \times 0.1 = 9.36$ lbf (41.6 N). The unbalanced force acting on the body along the plane is $(3/5)(90) - (4/5)(36) - 9.36 = 15.84$ lbf (70.46 N) downward. $F = (W/9)\,a = (90/g)\,a$; therefore, $a = 0.176\,g = 56.6$ ft/s^2 (1.725 m/s^2). In SI units, $F = ma = 70.46 = 40.8a$; and $a = 1.725$ m/s^2. The body is acted upon by constant forces and starts from rest; therefore, $v = \int_0^5 a\,dt$, and at the end of 5 s, the velocity would be 28.35 ft/s (8.91 m/s).

EXAMPLE 2. The force with which a rope acts on a body is equal and opposite to the force with which the body acts on the rope, and each is equal to the tension in the rope. In Fig. 3.1.60*a*, neglecting the weight of the pulley and the rope, the tension in the cord must be the force of 27 lbf. For the 18-lb mass, the unbalanced force is $27 - 18 = 9$ lbf in the upward direction, i.e., $27 - 18 = (18/g)a$, and $a = 16.1$ ft/s^2 upward. In Fig. 3.1.60*b* the 27-lb force is replaced by a 27-lb mass. The unbalanced force is still $27 - 18 = 9$ lbf, but it now acts on two masses so that $27 - 18 = (45/g)$ and $a = 6.44$ ft/s^2. The 18-lb mass is accelerated upward, and the 27-lb mass is accelerated downward. The tension in the rope is equal to 18 lbf plus the unbalanced force necessary to give it an upward acceleration of $g/5$ or $T = 18 + (18/g)(g/5) = 21.6$ lbf. The tension is also equal to 27 lbf less the unbalanced force necessary to give it a downward acceleration of $g/5$ or $T = 27 - (27/g) \times (g/5) = 21.6$ lbf.

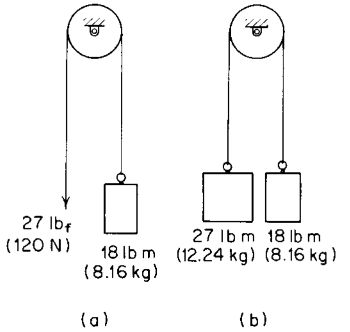


Fig. 3.1.60

In SI units, in Fig. 3.1.60*a*, the unbalanced force is $120 - 80 = 40$ N, in the upward direction, i.e., $120 - 80 = 8.16a$, and $a = 4.9$ m/s^2 (16.1 ft/s^2). In Fig. 3.1.60*b* the unbalanced force is still 40 N, but it now acts on the two masses so that $120 - 80 = 20.4a$ and $a = 1.96$ m/s^2 (6.44 ft/s^2). The tension in the rope is the weight of the 8.16-kg mass in newtons plus the unbalanced force necessary to give it an upward acceleration of 1.96 m/s^2, $T = 9.807(8.16) + (8.16)(1.96) = 96$ N (21.6 lbf).

General Formulas for the Motion of a Body under the Action of a Constant Unbalanced Force

Let s = space, ft; a = acceleration, ft/s^2; v = velocity, ft/s; v_0 = initial velocity, ft/s; h = height, ft; F = force; m = mass; w = weight; g = acceleration due to gravity.

Initial velocity = 0

$$F = ma = (w/g)a$$
$$v = at$$
$$s = \tfrac{1}{2}at^2 = \tfrac{1}{2}vt$$
$$v = \sqrt{2as}$$
$$= \sqrt{2gh} \text{ (falling freely from rest)}$$

Initial velocity = v

$$F = ma = (w/g)a$$
$$v = v_0 + at$$
$$s = v_0t + \tfrac{1}{2}at^2 = \tfrac{1}{2}v_0t + \tfrac{1}{2}vt$$

If a body is to be moved in a straight line by a force, the line of action of this force must pass through its center of gravity.

General Rule for the Solution of Problems When the Forces Are Constant in Magnitude and Direction

Resolve all the forces acting on the body into two components, one in the direction of the body's motion and one at right angles to it. Add the components in the direction of the body's motion algebraically and find the **unbalanced force,** if any exists.

In **curvilinear motion,** a particle moves along a curved path, and the resultant of the unbalanced force system may have components in directions other than the direction of motion. **The acceleration in any given direction is proportional to the component of the resultant in that direction.** It is common to utilize orthogonal coordinate systems such as **cartesian coordinates, polar coordinates,** and **normal and tangential coordinates** in analyzing forces and accelerations.

EXAMPLE. A conical pendulum consists of a weight suspended from a cord or light rod and made to rotate in a horizontal circle about a vertical axis with a constant angular velocity of N r/min. For any given constant speed of rotation, the angle θ, the radius r, and the height h will have fixed values. Looking at Fig. 3.1.61, we see that the forces in the vertical direction must be balanced, $T \cos \theta = w$. The forces in the direction normal to the circular path of rotation are unbalanced such that $T \sin \theta = (w/g)a_n = (w/g)\omega^2 r$. Substituting $r = l \sin \theta$ in this last equation gives the value of the tension in the cord $T = (w/g)l\omega^2$. Dividing the second equation by the first and substituting $\tan \theta = r/h$ yields the additional relation that $h = g/\omega^2$.

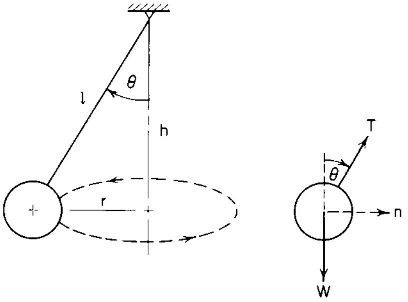


Fig. 3.1.61

An unresisted **projectile** has a motion compounded of the vertical motion of a falling body, and of the horizontal motion due to the horizontal component of the velocity of projection. In Fig. 3.1.62 the only force acting after the projectile starts is gravity, which causes an accelerating downward. The horizontal component of the original velocity v_0 is not changed by gravity. The projectile will rise until the velocity given to it by gravity is equal to the vertical component of the starting velocity v_0, and the equation $v_0 \sin \theta = gt$ gives the time t required to reach the highest point in the curve. The same time will be taken in falling if the surface XX is level, and the projectile will therefore be in flight $2t$ s. The distance $s = v_0 \cos \theta \times 2t$, and the maximum height of ascent $h = (v_0 \sin \theta)^2/2g$. The expressions for the coordinates of any point on the path of the projectile are: $x = (v_0 \cos \theta)t$, and $y = (v_0 \sin \theta)t - \frac{1}{2}gt^2$, giving $y = x \tan \theta - (gx^2/2v_0^2 \cos^2 \theta)$ as the equation for the curve of the path. The radius of curvature of the highest point may be found by using the general expression $v^2 = gr$ and solving for r, v being taken equal to $v_0 \cos \theta$.

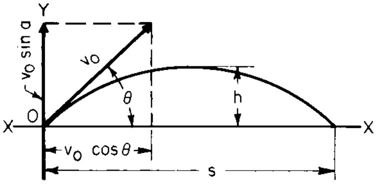


Fig. 3.1.62

Simple Pendulum The period of oscillation $= \tau = 2\pi\sqrt{l/g}$, where l is the length of the pendulum and the length of the swing is not great compared to l.

Centrifugal and Centripetal Forces When a body revolves about an axis, some connection must exist capable of applying force enough to the body to constantly deviate it toward the axis. This deviating force is known as **centripetal force.** The equal and opposite resistance offered by the body to the connection is called the **centrifugal force.** The acceleration toward the axis necessary to keep a particle moving in a circle about that axis is v^2/r; therefore, the force necessary is $ma = mv^2/r = wv^2/gr = w\pi^2N^2r/900g$, where N = r/min. This force is constantly directed toward the axis.

The centrifugal force of a solid body revolving about an axis is the same as if the whole mass of the body were concentrated at its center of gravity. Centrifugal force $= wv^2/gr = mv^2/r = w\omega^2 r/g$, where w and m are the weight and mass of the whole body, r is the distance from the axis about which the body is rotating to the center of gravity of the body, ω the angular velocity of the body about the axis in radians, and v the linear velocity of the center of gravity of the body.

Balancing

A rotating body is said to be in **standing balance** when its center of gravity coincides with the axis upon which it revolves. Standing balance may be obtained by resting the axis carrying the body upon two horizontal plane surfaces, as in Fig. 3.1.63. If the center of gravity of the wheel A coincides with the center of the shaft B, there will be no movement, but if the center of gravity does not coincide with the center of the shaft, the shaft will roll until the center of gravity of the wheel comes directly under the center of the shaft. The center of gravity may be brought to the center of the shaft by adding or taking away weight at proper points on the diameter passing through the center of gravity and the center of the shaft. Weights may be added to or subtracted from any part of the wheel so long as its center of gravity is brought to the center of the shaft.

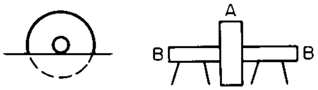


Fig. 3.1.63

A rotating body may be in standing balance and not in **dynamic balance.** In Fig. 3.1.64, AA and BB are two disks whose centers of gravity are at o and p, respectively. The shaft and the disks are in standing balance if the disks are of the same weight and the distances of o and p from the center of the shaft are equal, and o and p lie in the same axial plane but on opposite sides of the shaft. Let the weight of each disk be w and the distances of o and p from the center of the shaft each be equal to r. The force exerted on the shaft by AA is equal to $w\omega^2 r/g$, where ω is the angular velocity of the shaft. Also, the force exerted on the shaft by $BB = w\omega^2 r/g$. These two equal and opposite parallel forces act at a distance x apart and constitute a couple with a moment tending to rotate the shaft, as shown by the arrows, of $(w\omega^2 r/g)x$. A couple cannot be balanced by a single force; so two forces at least must be added to or subtracted from the system to get dynamic balance.

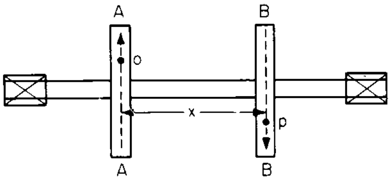


Fig. 3.1.64

Systems of Particles The principles of motion for a single particle can be extended to cover a **system of particles.** In this case, **the vector resultant of all external forces acting on the system of particles must equal the total mass of the system times the acceleration of the mass center, and the direction of the resultant must be the direction of the acceleration of the mass center.** This is the **principle of motion of the mass center.**

Rotation of Solid Bodies in a Plane about Fixed Axes

For a rigid body revolving **in a plane** about a fixed axis, **the resultant moment about that axis must be equal to the product of the moment of inertia (about that axis) and the angular acceleration,** $\Sigma M_0 = I_0\alpha$. This is a general statement which includes the particular case of rotation about an axis that passes through the center of gravity.

Rotation about an Axis Passing through the Center of Gravity The rotation of a body about its center of gravity can only be caused or changed by a **couple.** See Fig. 3.1.65. If a single force F is applied to the wheel, the axis immediately acts on the wheel with an equal force to prevent translation, and the result is a couple (moment Fr) acting on the body and causing rotation about its center of gravity.

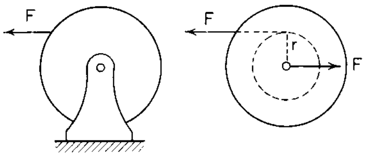


Fig. 3.1.65

General formulas for rotation of a body about a fixed axis through the center of gravity, if a constant unbalanced moment is applied (Fig. 3.1.65).

Let θ = angular displacement, rad; ω = angular velocity, rad/s; α = angular acceleration, rad/s²; M = unbalanced moment, ft · lb; I = moment of inertia (mass); g = acceleration due to gravity; t = time of application of M.

Initial angular velocity $= 0$	Initial angular velocity $= \omega_0$
$M = I\alpha$	$M = I\alpha$
$\theta = \frac{1}{2}\alpha t^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$\omega = \sqrt{2\alpha\theta}$	$\omega = \sqrt{\omega_0^2 + 2\alpha\theta}$

General Rule for Rotating Bodies Determine all the external forces acting and their moments about the axis of rotation. If these moments are balanced, there will be no change of motion. If the moments are unbalanced, this unbalanced moment, or **torque,** will cause an angular acceleration about the axis.

Rotation about an Axis Not Passing through the Center of Gravity **The resultant force acting on the body must be proportional to the acceleration of the center of gravity and directed along its line of action.** If the axis of rotation does not pass through the center of gravity, the center of gravity will have a resultant acceleration with a component $a_n = \omega^2 r$ directed toward the axis of rotation and a component $a_t = \alpha r$ tangential to its circular path. The resultant force acting on the body must also have two components, one directed normal and one directed tangential to the path of the center of gravity. The line of action of this resultant does not pass through the center of gravity because of the unbalanced moment $M_0 = I_0\alpha$ but at a point Q, as in Fig. 3.1.66. The point of application of this resultant is known as the **center of percussion** and may be **defined** as the point of application of the resultant of all the forces tending to cause a body to rotate about a certain axis. It is the point at which a suspended body may be struck without causing any force on the axis passing through the point of suspension.

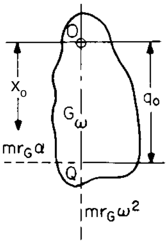


Fig. 3.1.66

Center of Percussion The distance from the axis of suspension to the center of percussion is $q_0 = I/mx_0$, where I = moment of inertia of the body about its axis of suspension to the center of gravity of the body.

EXAMPLES. 1. Find the center of percussion of the homogeneous rod (Fig. 3.1.67) of length L and mass m, suspended at XX.

$$q_0 = \frac{I}{mx_0}$$

$$I\,(\text{approx}) = \frac{m}{L}\int_0^L x^2\,dx \qquad x_0 = \frac{L}{2} \qquad \therefore q_0 = \frac{2}{L^2}\int_0^L x^2\,dx = 2L/3$$

2. Find the center of percussion of a solid cylinder, of mass m, resting on a horizontal plane. In Fig. 3.1.68, the instantaneous center of the cylinder is at A. The center of percussion will therefore be a height above the plane equal to $q_0 = I/mx_0$. Since $I = (mr^2/2) + mr^2$ and $x_0 = r$, $q_0 = 3r/2$.

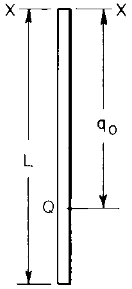


Fig. 3.1.67

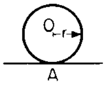


Fig. 3.1.68

Wheel or Cylinder Rolling down a Plane In this case the component of the weight along the plane tends to make it roll down and is treated as a force causing rotation. The forces acting on the body should be resolved into components along the line of motion and perpendicular to it. If the forces are all known, their resultant is at the center of percussion. If one force is to be determined (the exact conditions as regards slipping or not slipping must be known), the center of percussion can be determined and the unknown force found.

Relation between the Center of Percussion and Radius of Gyration $q_0 = I/mx_0 = k^2/x_0 \therefore k^2 = x_0 q_0$ where k = radius of gyration. Therefore, the radius of gyration is a mean proportional between the distance from the axis of oscillation to the center of percussion and the distance from the same axis to the center of gravity.

Interchangeability of Center of Percussion and Axis of Oscillation If a body is suspended from an axis, the center of percussion for that axis can be found. If the body is suspended from this center of percussion as an axis, the original axis of suspension will then become the center of percussion. The center of percussion is sometimes known as the **center of oscillation.**

Period of Oscillation of a Compound Pendulum The length of an equivalent simple pendulum is the distance from the axis of suspension to the center of percussion of the body in question. To find the **period of oscillation** of a body about a given axis, find the distance $q_0 = I/mx_0$ from that axis to the center of percussion of the swinging body. The length of the simple pendulum that will oscillate in the same time is this distance q_0. The period of oscillation for the equivalent single pendulum is $\tau = 2\pi\sqrt{q_0/g}$.

Determination of Moment of Inertia by Experiment To find the moment of inertia of a body, suspend it from some axis not passing through the center of gravity and, by swinging it, determine the period of one complete oscillation in seconds. The known values will then be τ = time of one complete oscillation, x_0 = distance from axis to center of gravity, and m = mass of body. The length of the equivalent simple pendulum is $q_0 = I/mx_0$. Substituting this value of q_0 in $\tau = 2\pi\sqrt{q_0/g}$ gives $\tau = 2\pi\sqrt{I/mx_0 g}$, from which $\tau^2 = 4\pi^2 I/mx_0 g$, or $I = mx_0 g\tau^2/4\pi^2$.

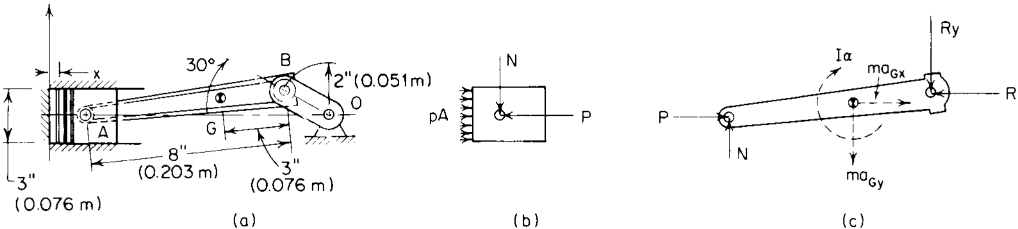


Fig. 3.1.69

Plane Motion of a Rigid Body

Plane motion may be considered to be a combination of translation and rotation (see ''Kinematics''). For translation, Newton's second law of motion must always be satisfied, and the resultant of the external force system must be equal to the product of the mass times the acceleration of the center of gravity in any system of coordinates. In rotation, the body moving in plane motion will not have a fixed axis. When the methods of relative motion are being used, any point on the body may be used as a reference axis to which the motion of all other points is referred.

The sum of the moments of all external forces about the reference axis must be equal to the vector sum of the centroidal moment of inertia times the angular acceleration and the amount of the resultant force about the reference axis.

EXAMPLE. Determine the forces acting on the piston pin *A* and the crankpin *B* of the connecting rod of a reciprocating engine shown in Fig. 3.1.69 for a position of 30° from TDC. The crankshaft speed is constant at 2,000 r/min. Assume that the pressure of expanding gases on the 4-lbm (1.81-kg) piston at this point is 145 lb/in^2 (10^6 N/m^2). The connecting rod has a mass of 5 lbm (2.27 kg) and has a centroidal radius of gyration of 3 in (0.076 m).

The kinematics of the problem are such that the angular velocity of the crank is ω_{OB} = 209.4 rad/s clockwise, the angular velocity of the connecting rod is ω_{AB} = 45.7 rad/s counterclockwise, and the angular acceleration is α_{AB} = 5,263 rad/s^2 clockwise. The linear acceleration of the piston is 7,274 ft/s^2 in the direction of the crank. From the free-body diagram of the piston, the horizontal component of the piston-pin force is $145 \times (\pi/4)(5^2) - P = (4/32.2)(7{,}274)$, $P = 1{,}943$ lbf. The acceleration of the center of gravity *G* is the vector sum of the component accelerations $a_G = a_B + a^{\text{n}}_{G/B} + a^{1}_{G/B}$ where $a^n_{G/B} = \omega^2_{GB} \cdot r_{GB} = 3/12(45.7)^2 = 522$ ft/s^2 and $a^{1}_{G/B} = \alpha_{GB} \cdot r_{GB} = 3/12(5.263) = 1{,}316$ ft/s^2. The resultant acceleration of the center of gravity is 6,685 ft/s^2 in the *x* direction and 2,284 ft/s^2 in the negative *y* direction. The resultant of the external force system will have corresponding components such that $ma_{Gx} = (5/32.2)(6{,}685) = 1{,}039$ lbf and $ma_{Gy} = (5/32.2)(2{,}284) = 355$ lbf. The three remaining unknown forces can be found from the three equations of motion for the connecting rod.

Taking the sum of the forces in the *x* direction, $\epsilon F = ma_{Gx}$; $P - R_x = ma_{Gx}$, and R_x = 905.4 lbf. In the *y* direction, $\Sigma F = ma_{Gy}$; $R_y - N = ma_{gy}$; this has two unknowns, R_y and *N*. Taking the sum of the moments of the external forces about the center of mass *g*, $\Sigma M_G = I_G \alpha AB$; $(N)(5)\cos(7.18°) - (P)(5)\sin(7.18°) + (R_y)(3)\cos(7.18°) - R_x(3)\sin(7.18°) - (5/386.4)(3)^2(5{,}263)$. Solving for R_y and *N* simultaneously, R_y = 494.7 lbf and N = 140 lbf. We could have avoided the solution of two simultaneous algebraic equations by taking the moment summation about end *A*, which would determine R_y independently, or about end *B*, which would determine *N* independently.

In SI units, the kinematics would be identical, the linear acceleration of the piston being 2,217 m/s^2 (7,274 ft/s^2). From the free-body diagram of the piston, the horizontal component of the piston-pin force is $(10^6) \times (\pi/4)(0.127)^2 - P = (1.81)(2{,}217)$, and $P = 8{,}640$ N. The components of the acceleration of the center of gravity *G* are $a^N_{G/B} = 522$ ft/s^2 and $a^T_{G/B} = 1{,}315$ ft/s^2. The resultant acceleration of the center of gravity is 2,037.5 m/s^2 (6,685 ft/s^2) in the *x* direction and 696.3 m/s^2 (2,284 ft/s^2) in the negative *y* direction. The resultant of the external force system will have the corresponding components; $ma_{Gx} = (2.27)(2{,}037.5) = 4{,}620$ N; $ma_{Gy} = (2.27)(696.3) = 1{,}579$ N. $R_x = 4{,}027$ N, $R_y = 2{,}201$ N, force $N = 623$ newtons.

WORK AND ENERGY

Work When a body is displaced against resistance or accelerated, work must be done upon it. An increment of work is defined as the product of an incremental displacement and the component of the force vector in the direction of the displacement or the product of the component of the incremental displacement and the force in the direction of the force. $dU = F \cdot ds \cos \alpha$, where α is the angle between the vector displacement and the vector force. The increment of work done by a couple *M* acting in a body during an increment of angular rotation $d\theta$ in the plane of the couple is $dU = M\, d\theta$. In a force-displacement or moment-angle diagram, called a **work diagram** (Fig. 3.1.70), force is plotted as a function of displacement. The area under the curve represents the work done, which is equal to $\int_{s_1}^{s_2} F\, ds \cos \alpha$ or $\int_{\theta_1}^{\theta_2} M\, d\theta$.

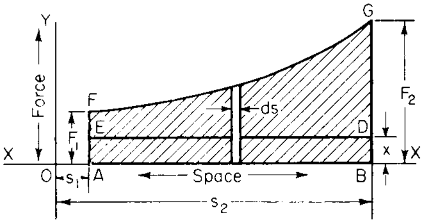


Fig. 3.1.70

Units of Work When the force of 1 lb acts through the distance of 1 ft, 1 lb · ft of work is done. In SI units, a force of 1 newton acting through 1 metre is 1 joule of work. 1.356 N · m = 1 lb · ft.

Energy A body is said to possess energy when it can do work. A body may possess this capacity through its **position** or **condition.** When a body is so held that it can do work, if released, it is said to possess energy of position or **potential energy.** When a body is moving with some velocity, it is said to possess energy of motion or **kinetic energy.** An example of potential energy is a body held suspended by a rope; the position of the body is such that if the rope is removed work can be done by the body.

Energy is expressed in the same units as work. The kinetic energy of a particle is expressed by the formula $E = \frac{1}{2}mv^2 = \frac{1}{2}(w/g)v^2$. The kinetic energy of a rigid body in translation is also expressed as $E = \frac{1}{2}mv^2$. Since all particles of the rigid body have the same identical velocity *v*, the velocity *v* is the velocity of the center of gravity. The kinetic energy of a rigid body, rotating about a fixed axis is $E = \frac{1}{2}I_0\omega^2$, where I_0 is the mass moment of inertia about the axis of rotation. In plane motion, a rigid body has both translation and rotation. The kinetic energy is the algebraic sum of the translating kinetic energy of the center of gravity and the rotating kinetic energy about the center of gravity, $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Here the velocity *v* is the velocity of the center of gravity, and the moment of inertia *I* is the centroidal moment of inertia.

If a force which varies acts through a space on a body of mass *m*, the work done is $\int_{s_1}^{s} F\, ds$, and if the work is all used in giving kinetic energy to the body it is equal to $\frac{1}{2}m(v_2^2 - v_1^2)$ = **change in kinetic energy,** where v_2 and v_1 are the velocities at distances s_2 and s_1, respectively. This is a specific statement of the law of conservation of energy. **The principle of conservation of energy requires that the mechanical energy of a system remain unchanged if it is subjected only to forces which depend on position or configuration.**

Certain problems in which the velocity of a body at any point in its straight-line path when acted upon by varying forces is required can be easily solved by the use of a **work diagram.**

In Fig. 3.1.70, let a body start from rest at *A* and be acted upon by a force that varies in accordance with the diagram *AFGBA*. Let the resistance to motion be a constant force $= x$. Find the velocity of the body at point *B*. The area *AFGBA* represents the work done upon the body and the area *AEDBA* ($=$ force $x \times$ distance *AB*) represents the work that must be done to overcome resistance. The difference of these areas, or *EFGDE*, will represent work done in excess of that required to overcome resistance, and consequently is equal to the increase in kinetic energy. Equating the work represented by the area *EFGDE* to $½wv^2/g$ and solving for v will give the required velocity at *B*. If the body did not start from rest, this area would represent the change in kinetic energy, and the velocity could be obtained by the formula: Work $= ½(w/g)(v_1^2 - v_0^2)$, v_1 being the required velocity.

General Rule for Rectilinear Motion Resolve each force acting on the body into components, one of which acts along the line of motion of the body and the other at right angles to the line of motion. Take the sum of all the components acting in the direction of the motion and multiply this sum by the distance moved through for constant forces. (Take the average force times distance for forces that vary.) This product will be the total work done upon the body. If there is no unbalanced component, there will be no change in kinetic energy and consequently no change in velocity. If there is an unbalanced component, the change in kinetic energy will be this unbalanced component multiplied by the distance moved through.

The **work done by a system of forces acting on a body** is equal to the algebraic sum of the work done by each force taken separately.

Power is the rate at which work is performed, or the number of units of work performed in unit time. In the English engineering system, the units of power are the horsepower, or 33,000 lb · ft/min = 550 lb · ft/s, and the kilowatt = 1.341 hp = 737.55 lb · ft/s. In SI units, the unit of power is the watt, which is 1 newton-metre per second or 1 joule per second.

Friction Brake In Fig. 3.1.71 a pulley revolves under the band and in the direction of the arrow, exerting a pull of T on the spring. The friction of the band on the rim of the pulley is $(T - w)$, where w is the weight attached to one end of the band. Let the pulley make N r/min; then the work done per minute against friction by the rim of the pulley is $2\pi RN(T - w)$, and the horsepower absorbed by brake $= 2\pi RN(T - w)/33{,}000$.

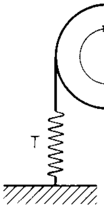


Fig. 3.1.71

IMPULSE AND MOMENTUM

The product of force and time is defined as linear impulse. The impulse of a constant force over a time interval $t_2 - t_1$ is $F(t_2 - t_1)$. If the force is not constant in magnitude but is constant in direction, the impulse is $\int_{t_1}^{t_2} F\,dt$. The dimensions of linear impulse are (force) × (time) in pound-seconds, or newton-seconds.

Impulse is a **vector** quantity which has the direction of the resultant force. Impulses may be added vectorially by means of a vector polygon, or they may be resolved into components by means of a parallelogram. The **moment** of a **linear impulse** may be found in the same manner as the moment of a force. The linear impulse is represented by a directed line segment, and the moment of the impulse is the product of the magnitude of the impulse and the perpendicular distance from the line segment to the point about which the moment is taken. **Angular impulse** over a time interval $t_2 - t_1$ is a product of the sum of applied moments on a rigid body about a reference axis and time. The dimensions for angular impulse are (force) × (time) × (displacement) in foot-pound-seconds or newton-metre-seconds. **Angular impulse and linear impulse cannot be added.**

Momentum is also a vector quantity and can be added and resolved in the same manner as force and impulse. The dimensions of linear momentum are (force) × (time) in pound-seconds or newton-seconds, and are identical to linear impulse. An alternate statement of Newton's second law of motion is that the resultant of an unbalanced force system must be equal to the time rate of change of linear momentum, $\Sigma\mathbf{F} = d(mv)/dt$.

If a variable force acts for a certain time on a body of mass m, the quantity $\int_{t_1}^{t_2} F\,dt = m(v_1 - v_2) =$ the change of momentum of the body.

The **moment of momentum** can be determined by the same methods as those used for the moment of a force or moment of an impulse. The dimensions of the moment of momentum are (force) × (time) × (displacement) in foot-pound-seconds, or newton-metre-seconds.

In **plane motion** the angular momentum of a rigid body about a reference axis perpendicular to the plane of motion is the sum of the moments of linear momenta of all particles in the body about the reference axes. Specifically, **the angular momentum of a rigid body in plane motion is the vector sum of the angular momentum about the reference axis and the moment of the linear momentum of the center of gravity about the reference axis,** $\mathbf{H}_0 = I_0\omega + \mathbf{d} \times mv$.

In three-dimensional rotation about a fixed axis, the angular momentum of a rigid body has components along three coordinate axes, which involve both the moments of inertia about the x, y, and z axes, $I_{0_{xx}}$, $I_{0_{yy}}$, and $I_{0_{zz}}$, and the products of inertia, $I_{0_{xy}}$, $I_{0_{zz}}$, and $I_{0_{yz}}$; $H_{0_x} - I_{0_{xx}} \cdot \omega_x - I_{0_{xy}} \cdot \omega_y - I_{0_{xy}} \cdot \omega_z$, $H_{0_y} = -I_{0_{xy}} \cdot \omega_x + I_{0_{yy}} \cdot \omega_y - I_{0_{yz}} \cdot \omega_z$, and $H_{0_z} = I_{0_{xz}} \cdot \omega_x - I_{0_{zy}} \cdot \omega_y + I_{0_{zz}} \cdot \omega_z$ where $\mathbf{H}_0 = \mathbf{H}_{0_x} + \mathbf{H}_{0_y} + \mathbf{H}_{0_z}$.

Impact

The collision between two bodies, where relatively large forces result over a comparatively short interval of time, is called **impact.** A straight line perpendicular to the plane of contact of two colliding bodies is called the **line of impact.** If the centers of gravity of the two bodies lie on the line of contact, the impact is called **central impact,** in any other case, **eccentric impact.** If the linear momenta of the centers of gravity are also directed along the line of impact, the impact is **collinear or direct central impact.** In any other case impact is said to be **oblique.**

Collinear Impact When two masses m_1 and m_2, having respective velocities u_1 and u_2, move in the same line, they will collide if $u_2 > u_1$ (Fig. 3.1.72*a*). During collision (Fig. 3.1.72*b*), kinetic energy is ab-

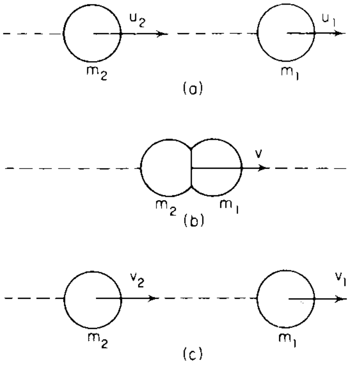


Fig. 3.1.72

sorbed in the deformation of the bodies. There follows a period of restoration which may or may not be complete. If complete restoration of the energy of deformation occurs, the impact is **elastic.** If the restoration of energy is incomplete, the impact is referred to as **inelastic.** After collision (Fig. 3.1.72*c*), the bodies continue to move with changed velocities of v_1 and v_2. Since the contact forces on one body are equal to and opposite the contact forces on the other, the sum of the linear momenta of the two bodies is conserved; $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$.

The law of conservation of momentum states that the linear momentum of a system of bodies is unchanged if there is no resultant external force on the system.

Coefficient of Restitution The ratio of the velocity of separation $v_1 - v_2$ to the velocity of approach $u_2 - u_1$ is called the **coefficient of restitution** e, $e = (v_1 - v_2)/(u_2 - u_1)$.

The value of e will depend on the shape and material properties of the colliding bodies. In elastic impact, the coefficient of restitution is unity and there is no energy loss. A coefficient of restitution of zero indicates perfectly inelastic or plastic impact, where there is no separation of the bodies after collision and the energy loss is a maximum. In **oblique impact,** the coefficient of restitution applies only to those components of velocity along the line of impact or normal to the plane of impact. The coefficient of restitution between two materials can be measured by making one body many times larger than the other so that m_2 is infinitely large in comparison to m_1. The velocity of m_2 is unchanged for all practical purposes during impact and $e = v_1/u_1$. For a small ball dropped from a height H upon an extensive horizontal surface and rebounding to a height h, $e = \sqrt{h/H}$.

Impact of Jet Water on Flat Plate When a jet of water strikes a flat plate perpendicularly to its surface, the force exerted by the water on the plate is wv/g, where w is the weight of water striking the plate in a unit of time and v is the velocity. When the jet is inclined to the surface by an angle, A, the pressure is (wv/g) cos A.

Variable Mass

If the mass of a body is variable such that mass is being either added or ejected, an alternate form of Newton's second law of motion must be used which accounts for changes in mass:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + \frac{dm}{dt}\mathbf{u}$$

The mass m is the instantaneous mass of the body, and dv/dt is the time rate of change of the absolute of velocity of mass m. The velocity u is the velocity of the mass m relative to the added or ejected mass, and dm/dt is the time rate of change of mass. In this case, care must be exercised in the choice of coordinates and expressions of sign. If mass is being added, dm/dt is plus, and if mass is ejected, dm/dt is minus.

Fields of Force—Attraction

The space within which the action of a physical force comes into play on bodies lying within its boundaries is called the **field of the force.**

The **strength** or **intensity of the field** at any given point is the relation between a force F acting on a mass m at that point and the mass. Intensity of field $= i = F/m$; $F = mi$.

The **unit of field intensity** is the same as the unit of acceleration, i.e., 1 ft/s^2 or 1 m/s^2. The intensity of a field of force may be represented by a line (or **vector**).

A field of force is said to be **homogeneous** when the intensity of all points is uniform and in the same direction.

A field of force is called a **central field of force** with a center O, if the direction of the force acting on the mass particle m in every point of the field passes through O and its magnitude is a function only of the distance r from O to m. A line so drawn through the field of force that its direction coincides at every point with that of the force prevailing at that point is called a **line of force.**

Rotation of Solid Bodies about Any Axis

The general moment equations for three-dimensional motion are usually expressed in terms of the angular momentum. For a reference axis O, which is either a fixed axis of the center of gravity, $M_{0_x} = (dH_{0_x}/dt) - H_{0_y} \cdot \omega_z + H_{0_z} \cdot \omega_y$, $M_{0_y} = (dH_{0_y}/dt) - H_{0_z} \cdot \omega_x + H_{0_x} \cdot \omega_z$, and $M_{0_z} = (dH_{0_z}/dt) - H_{0_x}\omega_y + H_{0_y}\omega_x$. If the coordinate axes are oriented to coincide with the principal axes of inertia, $I_{0_{xx}}$, $I_{0_{yy}}$, and $I_{0_{zz}}$, a similar set of three differential equations results, involving moments, angular velocity, and angular acceleration; $M_{0_x} = I_{0_{xx}}(d\omega_x/dt) + (I_{0_{zz}} - I_{0_{yy}})\omega_y \cdot \omega_z$, $M_{0_y} = I_{0_{yy}}(d\omega_y/dt) + (I_{0_{xx}} - I_{0_{zz}})\omega_z \cdot \omega_x$, and $M_{0_z} = I_{0_{zz}}(d\omega_z/dt) + (I_{0_{yy}} - I_{0_{xx}})\omega_x\omega_y$. These equations are known as **Euler's equations of motion** and may apply to any rigid body.

GYROSCOPIC MOTION AND THE GYROSCOPE

Gyroscopic motion can be explained in terms of Euler's equations. Let I_1, I_2, and I_3 represent the principal moments of inertia of a gyroscope spinning with a constant angular velocity ω, about axis 1, the subscripts 1, 2, and 3 representing a right-hand set of reference axes (Figs. 3.1.73 and 3.1.74). If the gyroscope is precessed about the third axis, a vector moment results along the second axis such that

$$M_2 = I_2\,(d\omega_2/dt) + (I_1 - I_3)\omega_3\omega_1$$

Where the precession and spin axes are at right angles, the term $(d\omega_2/dt)$ equals the component of $\omega_3 \times \omega_1$ along axis 2. Because of this, in the simple case of a body of symmetry, where $I_2 = I_3$, the gyroscopic moment can be reduced to the common expression $M = I\omega\Omega$, where Ω is the rate of precession, ω the rate of spin, and I the moment of inertia about the spin axis. It is important to realize that these are equations of motion and relate the applied or resulting gyroscopic moment due to forces which act *on* the rotor, as disclosed by a free-body diagram, to the resulting motion of the rotor.

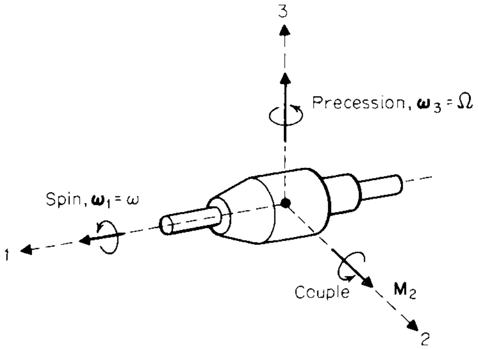


Fig. 3.1.73

Physical insight into the behavior of a steady precessing gyro with mutually perpendicular moment, spin, and precession axes is gained by recognizing from Fig. 3.1.74 that the change dH in angular momentum H is equal to the angular impulse $M\,dt$. In time dt, the angular-momen-

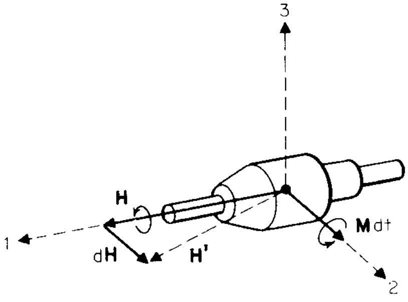


Fig. 3.1.74

tum vector swings from H to H', owing to the velocity of precession ω_3. The vector change dH in angular momentum is in the direction of the applied moment M. This fact is inherent in the basic moment-momentum equation and can always be used to establish the correct spatial relationships between the moment, precessional, and spin vectors. It is seen, therefore, from Fig. 3.1.74 that the spin axis always turns toward the moment axis. Just as the change in direction of the mass-center velocity is in the same direction as the resultant force, so does the change in angular momentum follow the direction of the applied moment.

For example, suppose an airplane is driven by a right-handed propeller (turning like a right-handed screw when moving forward). If a gust of wind or other force turns the machine to the *left*, the gyroscopic action of the propeller will make the forward end of the shaft strive to *rise;* if the wing surface is large, this motion will be practically prevented by the resistance of the air, and the gyroscopic forces become effective merely as internal stresses, whose maximum value can be computed by the formula above. Similarly, if the airplane is dipped *downward,* the gyroscopic action will make the forward end of the shaft strive to turn to the *left*.

Modern **applications** of the gyroscope are based on one of the following properties: (1) a gyroscope mounted in three gimbal rings so as to be entirely free angularly in all directions will retain its direction *in space* in the absence of outside couples; (2) if the axis of rotation of a gyroscope turns or precesses in space, a couple or torque acts on the gyroscope (and conversely on its frame).

Devices operating on the first principle are satisfactory only for short durations, say less than half an hour, because no gyroscope is entirely without outside couple. The friction couples at the various gimbal bearings, although small, will precess the axis of rotation so that after a while the axis of rotation will have changed its direction in space. The chief device based on the first principle is the **airplane compass,** which is a freely mounted gyro, keeping its direction in space during fast maneuvers of a fighting airplane. No magnetic compass will indicate correctly during such maneuvers. After the plane is back on an even keel in steady flight, the magnetic compass once more reads the true magnetic north, and the gyro compass has to be reset to point north again.

An example of a device operating on the second principle is the **automatic pilot** for keeping a vehicle on a given course. This device has been installed on torpedoes, ships, airplanes. When the ship or plane turns from the chosen course, a couple is exerted on the gyro axis, which makes it precess and this operates electric contacts or hydraulic or pneumatic valves. These again operate on the rudders, through relays, and bring the ship back to its course.

Another application is the ship **antirolling** gyroscope. This very large gyroscope spins about a vertical axis and is mounted in a ship so that the axis can be tipped fore and aft by means of an electric motor, the precession motor. The gyro can exert a large torque on the ship about the fore-and-aft axis, which is along the ‘‘rolling’’ axis. The sign of the torque is determined by the direction of rotation of the precession motor, which in turn is controlled by electric contacts operated by a small pilot gyroscope on the ship, which feels which way the ship rolls and gives the signals to apply a countertorque.

The **turn indicator** for airplanes is a gyro, the frame of which is held by springs. When the airplane turns, it makes the gyro axis turn with it, and the resultant couple is delivered by the springs. Thus the elongation of the springs is a measure of the rate of turn, which is suitably indicated by a pointer.

The most complicated and ingenious application of the gyroscope is the **marine compass.** This is a pendulously suspended gyroscope which is affected by gravity and also by the earth’s rotation so that the gyro axis is in equilibrium only when it points north, i.e., when it lies in the plane formed by the local vertical and by the earth’s north-south axis. If the compass is disturbed so that it points away from north, the action of the earth’s rotation will restore it to the correct north position in a few hours.

3.2 FRICTION

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REFERENCES: Bowden and Tabor, ‘‘The Friction and Lubrication of Solids,’’ Oxford. Fuller, ‘‘Theory and Practice of Lubrication for Engineers,’’ 2nd ed., Wiley. Shigley, ‘‘Mechanical Design,’’ McGraw-Hill. Rabinowicz, ‘‘Friction and Wear of Materials,’’ Wiley. Ling, Klaus, and Fein, ‘‘Boundary Lubrication—An Appraisal of World Literature,’’ ASME, 1969. Dowson, ‘‘History of Tribology,’’ Longman, 1979. Petersen and Winer, ‘‘Wear Control Handbook,’’ ASME, 1980.

Friction is the resistance that is encountered when two solid surfaces slide or tend to slide over each other. The surfaces may be either dry or lubricated. In the first case, when the surfaces are free from contaminating fluids, or films, the resistance is called **dry friction**. The friction of brake shoes on the rim of a railroad wheel is an example of dry friction.

When the rubbing surfaces are separated from each other by a very thin film of lubricant, the friction is that of **boundary** (or **greasy**) **lubrication**. The lubrication depends in this case on the strong adhesion of the lubricant to the material of the rubbing surfaces; the layers of lubricant slip over each other instead of the dry surfaces. A journal when starting, reversing, or turning at very low speed under a heavy load is an example of the condition that will cause boundary lubrication. Other examples are gear teeth (especially hypoid gears), cutting tools, wire-drawing dies, power screws, bridge trunnions, and the running-in process of most lubricated surfaces.

When the lubrication is arranged so that the rubbing surfaces are separated by a fluid film, and the load on the surfaces is carried entirely by the hydrostatic or hydrodynamic pressure in the film, the friction is that of **complete** (or **viscous**) **lubrication**. In this case, the frictional losses are due solely to the internal fluid friction in the film. Oil ring bearings, bearings with forced feed of oil, pivoted shoe-type thrust and journal bearings, bearings operating in an oil bath, hydrostatic oil pads, oil lifts, and step bearings are instances of complete lubrication.

Incomplete lubrication or mixed lubrication takes place when the load on the rubbing surfaces is carried partly by a fluid viscous film and partly by areas of boundary lubrication. The friction is intermediate between that of fluid and boundary lubrication. Incomplete lubrication exists in bearings with drop-feed, waste-packed, or wick-fed lubrication, or on parallel-surface bearings.

STATIC AND KINETIC COEFFICIENTS OF FRICTION

In the absence of friction, the resultant of the forces between the surfaces of two bodies pressing upon each other is normal to the surface of contact. With friction, the resultant deviates from the normal.

If one body is pressed against another by a force P, as in Fig. 3.2.1, the first body will not move, provided the angle a included between the line of action of the force and a normal to the surfaces in contact does not exceed a certain value which depends upon the nature of the surfaces. The reaction force R has the same magnitude and line of action as the force P. In Fig. 3.2.1, R is resolved into two components: a force N

normal to the surfaces in contact and a force F_r parallel to the surfaces in contact. From the above statement it follows that, for motion not to occur,

$$F_r = N \tan a_0 = Nf_0$$

where $f_0 = \tan a_0$ is called the **coefficient of friction of rest** (or of **static friction**) and a_0 is the **angle of friction at rest**.

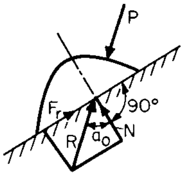


Fig. 3.2.1

If the normal force N between the surfaces is kept constant, and the tangential force F_r is gradually increased, there will be no motion while $F_r < Nf_0$. A state of *impending motion* is reached when F_r nears the value of Nf_0. If sliding motion occurs, a frictional force F resisting the motion must be overcome. The force F is commonly expressed as $F = fN$, where f is the **coefficient of sliding friction**, or **kinetic friction**. Normally, the coefficients of sliding friction are smaller than the coefficients of static friction. With small velocities of sliding and very clean surfaces, the two coefficients do not differ appreciably.

Table 3.2.4 demonstrates the typical reduction of sliding coefficients of friction below corresponding static values. Figure 3.2.2 indicates results of tests on lubricated machine tool ways showing a reduction of friction coefficient with increasing sliding velocity.

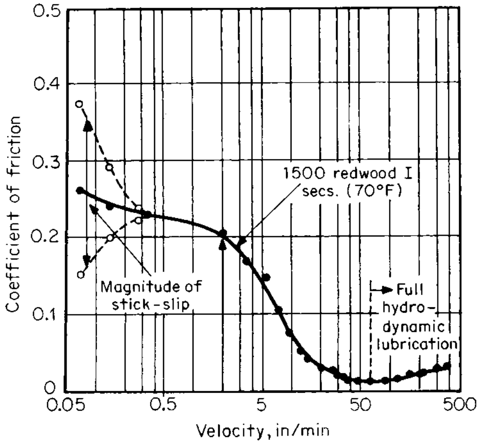


Fig. 3.2.2 Typical relationship between kinetic friction and sliding velocity for lubricated cast iron on cast iron slideways (load, 20 lb/in^2; upper slider, scraped; lower slideway, scraped). *(From Birchall, Kearny, and Moss, Intl. J. Machine Tool Design Research, 1962.)*

This behavior is normal with dry friction, some conditions of boundary friction, and with the break-away friction in ball and roller bearings. This condition is depicted in Fig. 3.2.3, where the friction force decreases with relative velocity. This negative slope leads to locally unstable equilibrium and **self-excited vibrations** in systems such as the one of Fig. 3.2.4. This phenomenon takes place because, for small amplitudes, the oscillatory system displays damping in which the damping factor is equal to the slope of the friction curve and thus is termed **negative damping**. When the slope of the friction force versus sliding velocity is positive (**positive damping**) this type of instability is not possible. This is typical of fluid damping, squeeze films, dash pots, and fluid film bearings in general.

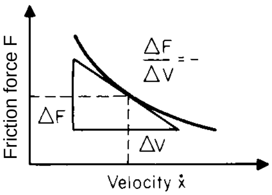


Fig. 3.2.3 Friction force decreases as velocity increases.

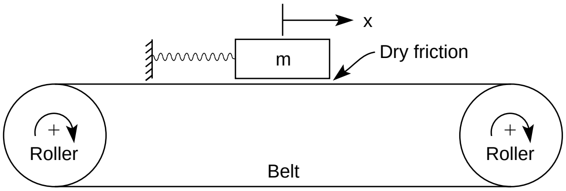


Fig. 3.2.4 Belt friction apparatus with possible self-excited vibrations.

It is interesting to note that these self-excited systems vibrate at close to their natural frequency over a large range of frictional levels and speeds. This symptom is a helpful means of identification. Another characteristic is that the moving body comes periodically to momentary relative rest, that is, zero sliding velocity. For this reason, this phenomenon is also called **stick-slip vibration**. Common examples are violin strings, chalk on blackboard, water-lubricated rubber stern tube ship bearings at low speed, squeaky hinges, and oscillating rolling element bearings, especially if they are supporting large flexible structures such as radar antennas. Control requires the introduction of fluid film bearings, viscous seals, or viscous dampers into the system with sufficient positive damping to override the effects of negative damping.

Under moderate pressures, the frictional force is proportional to the normal load on the rubbing surfaces. It is independent of the pressure per unit area of the surfaces. The direction of the friction force opposing the sliding motion is locally exactly opposite to the local relative velocity. Therefore, it takes very little effort to displace transversally two bodies which have a major direction of relative sliding. This behavior, **compound sliding,** is exploited when easing the extraction of a nail by simultaneously rotating it about its axis, and accounts for the ease with which an automobile may skid on the road or with which a plug gage can be inserted into a hole if it is rotated while being pushed in.

The coefficients of friction for dry surfaces (dry friction) depend on the materials sliding over each other and on the finished condition of the surfaces. With greasy (boundary) lubrication, the coefficients depend both on the materials and conditions of the surfaces and on the lubricants employed.

Coefficients of friction are sensitive to atmospheric dust and humidity, oxide films, surface finish, velocity of sliding, temperature, vibration, and the extent of contamination. In many instances the degree of contamination is perhaps the most important single variable. For example, in Table 3.2.1, values for the static coefficient of friction of steel on steel are listed, and, depending upon the degree of contamination of the specimens, the coefficient of friction varies effectively from ∞ (infinity) to 0.013.

The most effective boundary lubricants are generally those which react chemically with the solid surface and form an adhering film that is attached to the surface with a chemical bond. This action depends upon

Table 3.2.1 Coefficients of Static Friction for Steel on Steel

Test condition	f_0	Ref.
Degassed at elevated temp in high vacuum	∞ (weld on contact)	1
Grease-free in vacuum	0.78	2
Grease-free in air	0.39	3
Clean and coated with oleic acid	0.11	2
Clean and coated with solution of stearic acid	0.013	4

SOURCES: (1) Bowden and Young, *Proc. Roy. Soc.*, 1951. (2) Campbell, *Trans. ASME*, 1939. (3) Tomlinson, *Phil. Mag.*, 1929. (4) Hardy and Doubleday, *Proc. Roy. Soc.*, 1923.

the nature of the lubricant and upon the reactivity of the solid surface. Table 3.2.2 indicates that a fatty acid, such as found in animal, vegetable, and marine oils, reduces the coefficient of friction markedly only if it can react effectively with the solid surface. Paraffin oil is almost completely nonreactive.

Table 3.2.2 Coefficients of Static Friction at Room Temperature

Surfaces	Clean	Paraffin oil	Paraffin oil plus 1% lauric acid	Degree of reactivity of solid
Nickel	0.7	0.3	0.28	Low
Chromium	0.4	0.3	0.3	Low
Platinum	1.2	0.28	0.25	Low
Silver	1.4	0.8	0.7	Low
Glass	0.9		0.4	Low
Copper	1.4	0.3	0.08	High
Cadmium	0.5	0.45	0.05	High
Zinc	0.6	0.2	0.04	High
Magnesium	0.6	0.5	0.08	High
Iron	1.0	0.3	0.2	Mild
Aluminum	1.4	0.7	0.3	Mild

SOURCE: From Bowden and Tabor, ''The Friction and Lubrication of Solids,'' Oxford.

Values in Table 3.2.4 of sliding and static coefficients have been selected largely from investigations where these variables have been very carefully controlled. They are representative values for smooth surfaces. It has been generally observed that sliding friction between hard materials is smaller than that between softer surfaces.

Effect of Surface Films Campbell observed a lowering of the coefficient of friction when oxide or sulfide films were present on metal surfaces (*Trans. ASME*, 1939; footnotes to Table 3.2.4). The reductions listed in Table 3.2.3 were obtained with oxide films formed by heating in air at temperatures from 100 to 500° C, and sulfide films produced by immersion in a 0.02 percent sodium sulfide solution.

Table 3.2.3 Static Coefficient of Friction f_0

	Clean and dry	Oxide film	Sulfide film
Steel-steel	0.78	0.27	0.39
Brass-brass	0.88		0.57
Copper-copper	1.21	0.76	0.74

Effect of Sliding Velocity It has generally been observed that coefficients of friction reduce on dry surfaces as sliding velocity increases. (See results of railway brake-shoe tests below.) Dokos measured this reduction in friction for mild steel on medium steel. Values are for the average of four tests with high contact pressures (*Trans. ASME*, 1946; see footnotes to Table 3.2.4).

Sliding velocity, in/s	0.0001	0.001	0.01	0.1	1	10	100
f	0.53	0.48	0.39	0.31	0.23	0.19	0.18

Effect of Surface Finish The degree of surface roughness has been found to influence the coefficient of friction. Burwell evaluated this effect for conditions of boundary or greasy friction (*Jour. SAE*, 1942; see footnotes to Table 3.2.4). The values listed in Table 3.2.5 are for sliding coefficients of friction, hard steel on hard steel. The friction coefficient and wear rates of **polymers against metals** are often lowered by decreasing the surface roughness. This is particularly true of composites such as those with polytetrafluoroethylene (PTFE) which function through transfer to the counterface.

Solid Lubricants In certain applications solid lubricants are used successfully. Boyd and Robertson with pressures ranging from 50,000 to 400,000 lb/in^2 (344,700 to 2,757,000 kN/m^2) found sliding coefficients of friction *f* for hard steel on hard steel as follows: powdered mica, 0.305; powdered soapstone, 0.306; lead iodide, 0.071; silver sulfate, 0.054; graphite, 0.058; molybdenum disulfide, 0.033; tungsten disulfide, 0.037; stearic acid, 0.029 (*Trans. ASME*, 1945; see footnotes to Table 3.2.4).

Coefficients of Static Friction for Special Cases

Masonry and Earth Dry masonry on brickwork, 0.6–0.7; timber on polished stone, 0.40; iron on stone, 0.3 to 0.7; masonry on dry clay, 0.51; masonry on moist clay, 0.33.

Earth on Earth Dry sand, clay, mixed earth, 0.4 to 0.7; damp clay, 1.0; wet clay, 0.31; shingle and gravel, 0.8 to 1.1.

Natural Cork On cork, 0.59; on pine with grain, 0.49; on glass, 0.52; on dry steel, 0.45; on wet steel, 0.69; on hot steel, 0.64; on oiled steel, 0.45; water-soaked cork on steel, 0.56; oil-soaked cork on steel, 0.42.

Coefficients of Sliding Friction for Special Cases

Soapy Wood Lesley gives for wood on wood, copiously lubricated with tallow, stearine, and soft soap (as used in launching practice), a starting coefficient of friction equal to 0.036, diminishing to an average value of 0.019 for the first 50 ft of motion of the ship. Rennie gives 0.0385 for wood on wood, lubricated with soft soap, under a load of 56 lb/in^2.

Asbestos-Fabric Brake Material The coefficient of sliding friction *f* of asbestos fabric against a cast-iron brake drum, according to Taylor and Holt (*NBS*, 1940) is 0.35 to 0.40 when at normal temperature. It drops somewhat with rise in brake temperature up to 300°F (149°C). With a further increase in brake temperature from 300 to 500°F (149 to 260°C) the value of *f* may show an increase caused by disruption of the brake surface.

Steel Tires on Steel Rails (Galton)

Speed, mi/h	Start	6.8	13.5	27.3	40.9	54.4	60
Values of *f*	0.242	0.088	0.072	0.07	0.057	0.038	0.027

Railway Brake Shoes on Steel Tires Galton and Westinghouse give, for cast-iron brakes, the following values for *f*, which decrease rapidly with the speed of the rim; the coefficient *f* decreases also with time, as the temperature of the shoe increases.

Speed, mi/h	10	20	30	40	50	60
f, when brakes were applied	0.32	0.21	0.18	0.13	0.10	0.06
f, after 5 s	0.21	0.17	0.11	0.10	0.07	0.05
f, after 12 s		0.13	0.10	0.08	0.06	0.05

Schmidt and Schrader confirm the marked decrease in the coefficient of friction with the increase of rim speed. They also show an irregular slight decrease in the value of *f* with higher shoe pressure on the wheel, but they did not find the drop in friction after a prolonged application of the brakes. Their observations are as follows:

Speed, mi/h	20	30	40	50	60
Coefficient of friction	0.25	0.23	0.19	0.17	0.16

Friction of Steel on Polymers A useful list of friction coefficients between steel and various polymers is given in Table 3.2.6.

Grindstones The coefficient of friction between coarse-grained sandstone and cast iron is $f = 0.21$ to 0.24; for steel, 0.29; for wrought iron, 0.41 to 0.46, according as the stone is freshly trued or dull; for fine-grained sandstone (wet grinding) $f = 0.72$ for cast iron, 0.94 for steel, and 1.0 for wrought iron.

Honda and Yamada give $f = 0.28$ to 0.50 for carbon steel on emery, depending on the roughness of the wheel.

Table 3.2.4 Coefficients of Static and Sliding Friction
(Reference letters indicate the lubricant used; numbers in parentheses give the sources. See footnote.)

	Static		Sliding	
Materials	Dry	Greasy	Dry	Greasy
Hard steel on hard steel	0.78 (1)	0.11 (1, *a*)	0.42 (2)	0.029 (5, *h*)
		0.23 (1, *b*)		0.081 (5, *c*)
		0.15 (1, *c*)		0.080 (5, *i*)
		0.11 (1, *d*)		0.058 (5, *j*)
		0.0075 (18, *p*)		0.084 (5, *d*)
		0.0052 (18, *h*)		0.105 (5, *k*)
				0.096 (5, *l*)
				0.108 (5, *m*)
				0.12 (5, *a*)
Mild steel on mild steel	0.74 (19)		0.57 (3)	0.09 (3, *a*)
				0.19 (3, *u*)
Hard steel on graphite	0.21 (1)	0.09 (1, *a*)		
Hard steel on babbitt (ASTM No. 1)	0.70 (11)	0.23 (1, *b*)	0.33 (6)	0.16 (1, *b*)
		0.15 (1, *c*)		0.06 (1, *c*)
		0.08 (1, *d*)		0.11 (1, *d*)
		0.085 (1, *e*)		
Hard steel on babbitt (ASTM No. 8)	0.42 (11)	0.17 (1, *b*)	0.35 (11)	0.14 (1, *b*)
		0.11 (1, *c*)		0.065 (1, *c*)
		0.09 (1, *d*)		0.07 (1, *d*)
		0.08 (1, *e*)		0.08 (11, *h*)
Hard steel on babbitt (ASTM No. 10)		0.25 (1, *b*)		0.13 (1, *b*)
		0.12 (1, *c*)		0.06 (1, *c*)
		0.10 (1, *d*)		0.055 (1, *d*)
		0.11 (1, *e*)		
Mild steel on cadmium silver				0.097 (2, *f*)
Mild steel on phosphor bronze			0.34 (3)	0.173 (2, *f*)
Mild steel on copper lead				0.145 (2, *f*)
Mild steel on cast iron		0.183 (15, *c*)	0.23 (6)	0.133 (2, *f*)
Mild steel on lead	0.95 (11)	0.5 (1, *f*)	0.95 (11)	0.3 (11, *f*)
Nickel on mild steel			0.64 (3)	0.178 (3, *x*)
Aluminum on mild steel	0.61 (8)		0.47 93)	
Magnesium on mild steel			0.42 (3)	
Magnesium on magnesium	0.6 (22)	0.08 (22, *y*)		
Teflon on Teflon	0.04 (22)			0.04 (22, *f*)
Teflon on steel	0.04 (22)			0.04 (22, *f*)
Tungsten carbide on tungsten carbide	0.2 (22)	0.12 (22, *a*)		
Tungsten carbide on steel	0.5 (22)	0.08 (22, *a*)		
Tungsten carbide on copper	0.35 (23)			
Tungsten carbide on iron	0.8 (23)			
Bonded carbide on copper	0.35 (23)			
Bonded carbide on iron	0.8 (23)			
Cadmium on mild steel			0.46 (3)	
Copper on mild steel	0.53 (8)		0.36 (3)	0.18 (17, *a*)
Nickel on nickel	1.10 (16)		0.53 (3)	0.12 (3, *w*)
Brass on mild steel	0.51 (8)		0.44 (6)	
Brass on cast iron			0.30 (6)	
Zinc on cast iron	0.85 (16)		0.21 (7)	
Magnesium on cast iron			0.25 (7)	
Copper on cast iron	1.05 (16)		0.29 (7)	
Tin on cast iron			0.32 (7)	
Lead on cast iron			0.43 (7)	
Aluminum on aluminum	1.05 (16)		1.4 (3)	
Glass on glass	0.94 (8)	0.01 (10, *p*)	0.40 (3)	0.09 (3, *a*)
		0.005 (10, *q*)		0.116 (3, *v*)
Carbon on glass			0.18 (3)	
Garnet on mild steel			0.39 (3)	
Glass on nickel	0.78 (8)		0.56 (3)	

(*a*) Oleic acid; (*b*) Atlantic spindle oil (light mineral); (*c*) castor oil; (*d*) lard oil; (*e*) Atlantic spindle oil plus 2 percent oleic acid; (*f*) medium mineral oil; (*g*) medium mineral oil plus ½ percent oleic acid; (*h*) stearic acid; (*i*) grease (zinc oxide base); (*j*) graphite; (*k*) turbine oil plus 1 percent graphite; (*l*) turbine oil plus 1 percent stearic acid; (*m*) turbine oil (medium mineral); (*n*) olive oil; (*p*) palmitic acid; (*q*) ricinoleic acid; (*r*) dry soap; (*s*) lard; (*t*) water; (*u*) rape oil; (*v*) 3-in-1 oil; (*w*) octyl alcohol; (*x*) triolein; (*y*) 1 percent lauric acid in paraffin oil.

SOURCES: (1) Campbell, *Trans. ASME*, 1939; (2) Clarke, Lincoln, and Sterrett, *Proc. API*, 1935; (3) Beare and Bowden, *Phil. Trans. Roy. Soc.*, 1935; (4) Dokos, *Trans. ASME*, 1946; (5) Boyd and Robertson, *Trans. ASME*, 1945; (6) Sachs, *Zeit f. angew. Math. und Mech.*, 1924; (7) Honda and Yamaha, *Jour. I of M*, 1925; (8) Tomlinson, *Phil. Mag.*, 1929; (9) Morin, *Acad. Roy. des Sciences*, 1838; (10) Claypoole, *Trans. ASME*, 1943; (11) Tabor, *Jour. Applied Phys.*, 1945; (12) Eyssen, General Discussion on Lubrication, *ASME*, 1937; (13) Brazier and Holland-Bowyer, General Discussion on Lubrication, *ASME*, 1937; (14) Burwell, *Jour. SAE.*, 1942; (15) Stanton, ''Friction,'' Longmans; (16) Ernst and Merchant, Conference on Friction and Surface Finish, M.I.T., 1940; (17) Gongwer, Conference on Friction and Surface Finish, M.I.T., 1940; (18) Hardy and Bircumshaw, *Proc. Roy. Soc.*, 1925; (19) Hardy and Hardy, *Phil. Mag.*, 1919; (20) Bowden and Young, *Proc. Roy. Soc.*, 1951; (21) Hardy and Doubleday, *Proc. Roy. Soc.*, 1923; (22) Bowden and Tabor, ''The Friction and Lubrication of Solids,'' Oxford; (23) Shooter, *Research*, **4,** 1951.

Table 3.2.4 Coefficients of Static and Sliding Friction (*Continued*)
(Reference letters indicate the lubricant used; numbers in parentheses give the sources. See footnote.)

Materials	Static		Sliding	
	Dry	Greasy	Dry	Greasy
Copper on glass	0.68 (8)		0.53 (3)	
Cast iron on cast iron	1.10 (16)		0.15 (9)	0.070 (9, *d*) 0.064 (9, *n*)
Bronze on cast iron			0.22 (9)	0.077 (9, *n*)
Oak on oak (parallel to grain)	0.62 (9)		0.48 (9)	0.164 (9, *r*) 0.067 (9, *s*)
Oak on oak (perpendicular)	0.54 (9)		0.32 (9)	0.072 (9, *s*)
Leather on oak (parallel)	0.61 (9)		0.52 (9)	
Cast iron on oak			0.49 (9)	0.075 (9, *n*)
Leather on cast iron			0.56 (9)	0.36 (9, *t*) 0.13 (9, *n*)
Laminated plastic on steel			0.35 (12)	0.05 (12, *t*)
Fluted rubber bearing on steel				0.05 (13, *t*)

(*a*) Oleic acid; (*b*) Atlantic spindle oil (light mineral); (*c*) castor oil; (*d*) lard oil; (*e*) Atlantic spindle oil plus 2 percent oleic acid; (*f*) medium mineral oil; (*g*) medium mineral oil plus ½ percent oleic acid; (*h*) stearic acid; (*i*) grease (zinc oxide base); (*j*) graphite; (*k*) turbine oil plus 1 percent graphite; (*l*) turbine oil plus 1 percent stearic acid; (*m*) turbine oil (medium mineral); (*n*) olive oil; (*p*) palmitic acid; (*q*) ricinoleic acid; (*r*) dry soap; (*s*) lard; (*t*) water; (*u*) rape oil; (*v*) 3-in-1 oil; (*w*) octyl alcohol; (*x*) triolein; (*y*) 1 percent lauric acid in paraffin oil.

SOURCES: (1) Campbell, *Trans. ASME,* 1939; (2) Clarke, Lincoln, and Sterrett, *Proc. API,* 1935; (3) Beare and Bowden, *Phil. Trans. Roy. Soc.,* 1935; (4) Dokos, *Trans. ASME,* 1946; (5) Boyd and Robertson, *Trans. ASME,* 1945; (6) Sachs, *Zeit f. angew. Math. und Mech.,* 1924; (7) Honda and Yamaha, *Jour. I of M,* 1925; (8) Tomlinson, *Phil. Mag.,* 1929; (9) Morin, *Acad. Roy. des Sciences,* 1838; (10) Claypoole, *Trans. ASME,* 1943; (11) Tabor, *Jour. Applied Phys.,* 1945; (12) Eyssen, General Discussion on Lubrication, *ASME,* 1937; (13) Brazier and Holland-Bowyer, General Discussion on Lubrication, *ASME,* 1937; (14) Burwell, *Jour. SAE.,* 1942; (15) Stanton, ‘‘Friction,’’ Longmans; (16) Ernst and Merchant, Conference on Friction and Surface Finish, M.I.T., 1940; (17) Gongwer, Conference on Friction and Surface Finish, M.I.T., 1940; (18) Hardy and Bircumshaw, *Proc. Roy. Soc.,* 1925; (19) Hardy and Hardy, *Phil. Mag.,* 1919; (20) Bowden and Young, *Proc. Roy. Soc.,* 1951; (21) Hardy and Doubleday, *Proc. Roy. Soc.,* 1923; (22) Bowden and Tabor, ‘‘The Friction and Lubrication of Solids,’’ Oxford; (23) Shooter, *Research,* **4,** 1951.

Table 3.2.5 Coefficient of Friction of Hard Steel on Hard Steel

	Surface					
	Superfinished	Ground	Ground	Ground	Ground	Grit-blasted
Roughness, microinches	2	7	20	50	65	55
Mineral oil	0.128	0.189	0.360	0.372	0.378	0.212
Mineral oil + 2% oleic acid	0.116	0.170	0.249	0.261	0.230	0.164
Oleic acid	0.099	0.163	0.195	0.222	0.238	0.195
Mineral oil + 2% sulfonated sperm oil	0.095	0.137	0.175	0.251	0.197	0.165

Table 3.2.6 Coefficient of Friction of Steel on Polymers
Room temperature, low speeds.

Material	Condition	*f*
Nylon	Dry	0.4
Nylon	Wet with water	0.15
Plexiglas	Dry	0.5
Polyvinyl chloride (PVC)	Dry	0.5
Polystyrene	Dry	0.5
Low-density (LD) polyethylene, no plasticizer	Dry	0.4
LD polyethylene, no plasticizer	Wet	0.1
High-density (HD) polyethylene, no plasticizer	Dry or wet	0.15
Soft wood	Natural	0.25
Lignum vitae	Natural	0.1
PTFE, low speed	Dry or wet	0.06
PTFE, high speed	Dry or wet	0.3
Filled PTFE (15% glass fiber)	Dry	0.12
Filled PTFE (15% graphite)	Dry	0.09
Filled PTFE (60% bronze)	Dry	0.09
Polyurethane rubber	Dry	1.6
Isoprene rubber	Dry	3–10
Isoprene rubber	Wet (water and alcohol)	2–4

Rubber Tires on Pavement Arnoux gives $f = 0.67$ for dry macadam, 0.71 for dry asphalt, and 0.17 to 0.06 for soft, slippery roads. For a cord tire on a sand-filled brick surface in fair condition. Agg (*Bull.* 88, *Iowa State College Engineering Experiment Station*, 1928) gives the following values of *f* depending on the inflation of the tire:

Inflation pressure, lb/in^2	Dry pavement		Wet pavement	
	Static f_0	Sliding f	Static f_0	Sliding f
40	0.90	0.85	0.74	0.69
50	0.88	0.84	0.64	0.58
60	0.80	0.76	0.63	0.56

Tests of the Goodrich Company on wet brick pavement with tires of different treads gave the following values of *f*:

	Coefficients of friction			
	Static (before slipping)		Sliding (after slipping)	
Speed, mi/h	5	30	5	30
Smooth tire	0.49	0.28	0.43	0.26
Circumferential grooves	0.58	0.42	0.52	0.36
Angular grooves at 60°	0.75	0.55	0.70	0.39
Angular grooves at 45°	0.77	0.55	0.68	0.44

Development continues using various manufacturing techniques (bias ply, belted, radial, studs), tread patterns, and rubber compounds, so that it is not possible to present average values applicable to present conditions.

Sleds For **unshod wooden runners** on smooth wood or stone surfaces, $f = 0.07$ (0.15) when tallow (dry soap) is used as a lubricant ($= 0.38$ when not lubricated); on snow and ice, $f = 0.035$. For **runners with metal**

shoes on snow and ice, $f = 0.02$. Rennie found for steel on ice, $f = 0.014$. However, as the temperature falls, the coefficient of friction will get larger. Bowden cites the following data for brass on ice:

Temperature, °C	f
0	0.025
− 20	0.085
− 40	0.115
− 60	0.14

ROLLING FRICTION

Rolling is substituted frequently for sliding friction, as in the case of wheels under vehicles, balls or rollers in bearings, rollers under skids when moving loads; frictional resistance to the rolling motion is substantially smaller than to sliding motion. The fact that a resistance arises to rolling motion is due to several factors: (1) the contacting surfaces are elastically deflected, so that, on the finite size of the contact, relative sliding occurs, (2) the deflected surfaces dissipate energy due to internal friction (hysteresis), (3) the surfaces are imperfect so that contact takes place on asperities ahead of the line of centers, and (4) surface adhesion phenomena. The coefficient of rolling friction $f_r = P/L$ where L is the load and P is the frictional resistance.

The frictional resistance P to the rolling of a cylinder under a load L applied at the center of the roller (Fig. 3.2.5) is inversely proportional to the radius r of the roller; $P = (k/r)L$. Note that k has the dimensions of length. Quite often k increases with load, particularly for cases involving plastic deformations. Values of k, in inches, are as follows: hardwood on hardwood, 0.02; iron on iron, steel on steel, 0.002; hard polished steel on hard polished steel, 0.0002 to 0.0004.

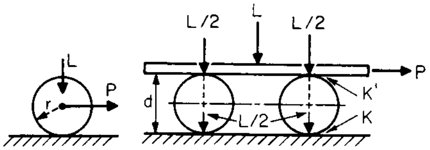


Fig. 3.2.5

Data on rolling friction are scarce. Noonan and Strange give, for steel rollers on steel plates and for loads varying from light to those causing a permanent set of the material, the following values of k, in inches: surfaces well finished and clean, 0.0005 to 0.001; surfaces well oiled, 0.001 to 0.002; surfaces covered with silt, 0.003 to 0.005; surfaces rusty, 0.005 to 0.01.

If a load L is moved on rollers (Fig. 3.2.5) and if k and k' are the respective coefficients of friction for the lower and upper surfaces, the frictional force $P = (k + k')L/d$.

McKibben and Davidson (*Agri. Eng.,* 1939) give the data in Table 3.2.7 on the rolling resistance of various types of wheels for typical road and field conditions. Note that the coefficient f_r is the ratio of resistance force to load.

Moyer found the following average values of f_r for pneumatic rubber tires properly inflated and loaded: hard road, 0.008; dry, firm, and well-packed gravel, 0.012; wet loose gravel, 0.06.

FRICTION OF MACHINE ELEMENTS

Work of Friction—Efficiency In a simple machine or assemblage of two elements, the work done by an applied force P acting through the distance s is measured by the product Ps. The useful work done is less and is measured by the product Ll of the resistance L by the distance l through which it acts. The **efficiency *e*** of the machine is the ratio of the useful work performed to the total work received, or $e = Ll/Ps$. The **work expended in friction** W_f is the difference between the total work received and the useful work, or $W_f = Ps - Ll$. The lost-work ratio $= V = W_f/Ll$, and $e = 1/(1 + V)$.

If a machine consists of a train of mechanisms having the respective efficiencies $e_1, e_2, e_3 \ . \ . \ . \ e_n$, the combined efficiency of the machine is equal to the product of these efficiencies.

Efficiencies of Machines and Machine Elements The values for machine elements in Table 3.2.8 are from ‘‘Elements of Machine Design,’’ by Kimball and Barr. Those for machines are from Goodman’s ‘‘Mechanics Applied to Engineering.’’ The quantities given are percentage efficiencies.

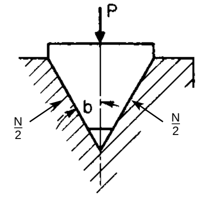


Fig. 3.2.6

Table 3.2.7 Coefficients of Rolling Friction f_r for Wheels with Steel and Pneumatic Tires

Wheel	Inflation press, lb/in^2	Load, lb	Concrete	Bluegrass sod	Tilled loam	Loose sand	Loose snow 10–14 in deep
2.5 × 36 steel		1,000	0.010	0.087	0.384	0.431	0.106
4 × 24 steel		500	0.034	0.082	0.468	0.504	0.282
4.00–18 4-ply	20	500	0.034	0.058	0.366	0.392	0.210
4 × 36 steel		1,000	0.019	0.074	0.367	0.413	
4.00–30 4-ply	36	1,000	0.018	0.057	0.322	0.319	
4.00–36 4-ply	36	1,000	0.017	0.050	0.294	0.277	
5.00–16 4-ply	32	1,000	0.031	0.062	0.388	0.460	
6 × 28 steel		1,000	0.023	0.094	0.368	0.477	0.156
6.00–16 4-ply	20	1,000	0.027	0.060	0.319	0.338	0.146
6.00–16 4-ply*	30	1,000	0.031	0.070	0.401	0.387	
7.50–10 4-ply†	20	1,000	0.029	0.061	0.379	0.429	
7.50–16 4-ply	20	1,500	0.023	0.055	0.280	0.322	
7.50–28 4-ply	16	1,500	0.026	0.052	0.197	0.205	
8 × 48 steel		1,500	0.013	0.065	0.236	0.264	0.118
7.50–36 4-ply	16	1,500	0.018	0.046	0.185	0.177	0.0753
9.00–10 4-ply†	20	1,000	0.031	0.060	0.331	0.388	
9.00–16 6-ply	16	1,500	0.042	0.054	0.249	0.272	0.099

* Skid-ring tractor tire.

† Ribbed tread tractor tire.

All other pneumatic tires with implement-type tread.

Wedges

Sliding in V Guides If a wedge-shaped slide having an angle $2b$ is pressed into a V guide by a force P (Fig. 3.2.6), the total force normal to the wedge faces will be $N = P/\sin b$. A friction force F, opposing motion along the longitudinal axis of the wedge, arises by virtue of the coefficient of friction f between the contacting surface of the wedge and guides: $F = fN = fP/\sin b$. In these formulas, the fact that the elasticity of the materials permits an advance of the wedge into the guide under the load P has been neglected. The common efficiency for V guides is $e = 0.88$ to 0.90.

Taper Keys In Fig. 3.2.7 if the key is moved in the direction of the force P, the force H must be overcome. The supporting reactions K_1, K_2, and K_3 together with the required force P may be obtained by drawing the force polygon (Fig. 3.2.8). The friction angles of these faces are a_1, a_2, and a_3, respectively. In Fig. 3.2.8, draw AB parallel to H in Fig. 3.2.7, and lay it off to scale to represent H. From the point A, draw AC parallel to K_1, i.e., making the angle $b + a_1$ with AB; from the other extremity of AB, draw BC parallel to K_2 in Fig. 3.2.7. AC and CB then give the magnitudes of K_1 and K_2, respectively. Now through C draw CD parallel to K_3 to its intersection with AD which has been drawn through A parallel to P. The magnitudes of K_3 and P are then given by the lengths of CD and DA.

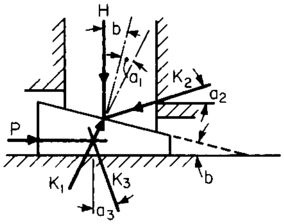


Fig. 3.2.7

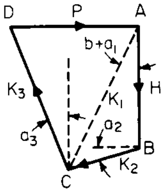


Fig. 3.2.8

By calculation,

$$K_1/H = \cos a_2/\cos (b + a_1 + a_2)$$
$$P/K_1 = \sin (b + a_1 + a_3)/\cos a_3$$
$$P/H = \cos a_2 \sin (b + a_1 + a_3)/\cos a_3 \cos (b + a_1 + a_2)$$

If $a_1 = a_2 = a_3 = a$, then $P = H \tan (b + 2a)$, and efficiency $e = \tan b/\tan (b + 2a)$. Force required to loosen the key $= P_1 = H \tan (2a - b)$. In order for the key not to slide out when force P is removed, it is necessary that $b < (a_1 + a_3)$, or $b < 2a$.

The forces acting upon the taper key of Fig. 3.2.9 may be found in a similar way (see Fig. 3.2.10).

$$\begin{aligned} P &= 2H \cos a \sin (b + a)/\cos (b + 2a) \\ &= 2H \tan (b + a)/[1 - \tan a \tan (b + a)] \\ &= 2H \tan (b + a) \text{ approx} \end{aligned}$$

The force to loosen the key is $P_1 = 2H \tan (a - b)$ approx, and the efficiency $e = \tan b/\tan (b + a)$. The key will be self-locking when $b < a$, or, more generally, when $2b < (a_1 + a_3)$.

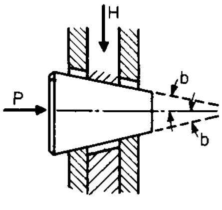


Fig. 3.2.9

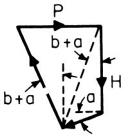


Fig. 3.2.10

Screws

Screws with Square Threads (Fig. 3.2.11) Let r = mean radius of the thread = ½ (radius at root + outside radius), and l = pitch (or lead of a single-threaded screw), both in inches; b = angle of inclination of thread to a plane at right angles to the axis of screw ($\tan b = l/2\pi r$); and f = coefficient of sliding friction = $\tan a$. Then for a screw in uniform motion (friction of the root and outside surfaces being neglected) there is required a force P acting at right angles to the axis at the distance r. $P = L \tan (b \pm a) = L(l \pm 2\pi rf)/(2\pi r \pm fl)$, where the upper signs are for motion in a direction opposed to that of L and the lower for motion in the same direction as that of L. When $b \leq a$, the screw will not ''overhaul'' (or move under the action of the load L).

The **efficiency** for motion opposed to direction in which L acts $= e = \tan b/\tan (b + a)$; for motion in the same direction in which L acts, $e = \tan (b - a)/\tan b$.

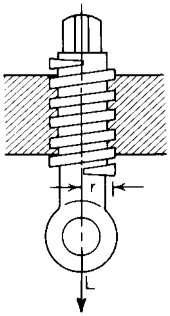


Fig. 3.2.11

The value of e is a maximum when $b = 45° - ½a$; e.g., $e_{max} = 0.81$ for $b = 42°$ and $f = 0.1$. Since e increases rapidly for values of b up to 20°, this angle is generally not exceeded; for $b = 20°$, and $f_1 = 0.10$, $e = 0.74$. In presses, where the mechanical advantage is required to be great, b is taken down to 3°, for which value $e = 0.34$ with $f = 0.10$.

Kingsbury found for square-threaded screws running in loose-fitting nuts, the following coefficients of friction: lard oil, 0.09 to 0.25; heavy mineral oil, 0.11 to 0.19; heavy oil with graphite, 0.03 to 0.15.

Ham and Ryan give for screws the following values of coefficients of friction, with medium mineral oil: high-grade materials and workmanship, 0.10; average quality materials and workmanship, 0.12; poor workmanship, 0.15. The use of castor oil as a lubricant lowered f from 0.10 to 0.066. The coefficients of static friction (at starting) were 30 percent higher. Table 3.2.8 gives representative values of efficiency.

Screws with V Threads (Fig.3.2.12) Let c = half the angle between the faces of a thread. Then, using the same notation as for square-threaded screws, for a screw in motion (neglecting friction of root and outside surfaces),

$$P = L(l \pm 2\pi rf \sec d)/(2\pi r \pm lf \sec d)$$

d is the angle between a plane normal to the axis of the screw through the point of the resultant thread friction, and a plane which is tangent to

Table 3.2.8 Efficiencies of Machines and Machine Elements

Common bearing (singly)	96–98
Common bearing, long lines of shafting	95
Roller bearings	98
Ball bearings	99
Spur gear, including bearings	
Cast teeth	93
Cut teeth	96
Bevel gear, including bearings	
Cast teeth	92
Cut teeth	95
Worm gear	
Thread angle, 30°	85–95
Thread angle, 15°	75–90
Belting	96–98
Pin-connected chains (bicycle)	95–97
High-grade transmission chains	97–99
Weston pulley block (½ ton)	30–47
Epicycloidal pulley block	40–45
1-ton steam hoist or windlass	50–70
Hydraulic windlass	60–80
Hydraulic jack	80–90
Cranes (steam)	60–70
Overhead traveling cranes	30–50
Locomotives (drawbar hp/ihp)	65–75
Hydraulic couplings, max	98

the surface of the thread at the same point (see Groat, *Proc. Engs. Soc. West. Penn,* **34**). Sec $d = \sec c \sqrt{1 - (\sin b \sin c)^2}$. For small values of b this reduces practically to $\sec d = \sec c$, and, for all cases the approximation, $P = L(l \pm 2\pi rf \sec c)/(2\pi r \pm lf \sec c)$ is within the limits of probable error in estimating values to be used for f.

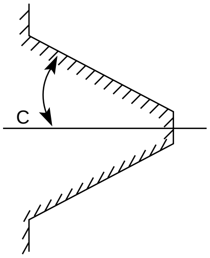


Fig. 3.2.12

The **efficiencies** are: $e = \tan b(1 - f \tan b \sec d)/(\tan b + f \sec d)$ for motion opposed to L, and $e = (\tan b - f \sec d)/\tan b(1 + f \tan b \sec d)$ for motion with L. If we let $\tan d' = f \sec d$, these equations reduce, respectively, to $e = \tan b/\tan (b + d')$ and $e = \tan (b - d')/\tan b$. Negative values in the latter case merely mean that the thread will not overhaul. Subtract the values from unity for actual efficiency, considering the external moment and not the load L as being the driver. The efficiency of a V thread is lower than that of a square thread of the same helix angle, since $d' > a$.

For a **V-threaded screw and nut,** let r_1 = outside radius of thread, r_2 = radius at root of thread, $r = (r_1 + r_2)/2$, $\tan d' = f \sec d$, r_0 = mean radius of nut seat = $1.5r$ (approx) and f' = coefficient of friction between nut and seat.

To tighten up the nut the turning moment required is $M = Pr + Lr_0 f = Lr[\tan (d' + b) + 1.5f']$. **To loosen** $M = Lr[\tan (d' - b) + 1.5f']$.

The **total tension in a bolt** due to tightening up with a moment M is $T = 2\pi M/(l + fl \sec b \sec d \operatorname{cosec} b + f'3\pi r)$. $T \div$ area at root gives unit pure tensile stress induced, S_t. There is also a unit torsional stress: $S_s = 2(M - 1.5rf'T)/\pi r_2^3$. The equivalent combined stress is $S = 0.35S_t + 0.65 \sqrt{S_t^2 + 4S_s^2}$.

Kingsbury, from tests on U.S. standard bolts, finds efficiencies for tightening up nuts from 0.06 to 0.12, depending upon the roughness of the contact surfaces and the character of the lubrication.

Toothed and Worm Gearing

The efficiency of **spur and bevel gearing** depends on the material and the workmanship of the gears and on the lubricant employed. For high-speed gears of good quality the efficiency of the gear transmission is 99 percent; with slow-speed gears of average workmanship the efficiency of 96 percent is common. On the average, efficiencies of 97 to 98 percent can be considered normal.

In **helical gears,** where considerable transverse sliding of the meshing teeth on each other takes place, the friction is much greater. If b and c are, respectively, the spiral angles of the teeth of the driving and driven helical gears (i.e., the angle between the teeth and the axis of rotation), $b + c$ is the shaft angle of the two gears, and $f = \tan a$ is the coefficient of sliding friction of the teeth, the efficiency of the gear transmission is $e = [\cos b \cos (c + a)]/[\cos c \cos (b - a)]$.

In the case of **worm gearing** when the shafts are normal to each other ($b + c = 90$), the efficiency is $e = \tan c/\tan (c + a) = (1 - pf/2\pi r)/(1 + 2\pi rf/p)$, where c is the spiral angle of the worm wheel, or the lead angle of the worm; p the lead, or pitch of the worm thread; and r the mean radius of the worm. Typical values of f are shown in Table 3.2.9.

Journals and Bearings

Friction of Journal Bearings If P = total load on journal, l = journal length, and $2r$ = journal diameter, then $p = P/2rl$ = mean normal pressure on the projected area of the journal. Also, if f_1 is the **coefficient of journal friction,** the **moment of journal friction** for a cylindrical journal is $M = f_1 Pr$. The **work expended in friction** at angular velocity ω is

$$W_f = \omega M = f_1 Pr\omega$$

For the **conical bearing** (Fig. 3.2.13) the mean radius $r_m = (r + R)/2$ is to be used.

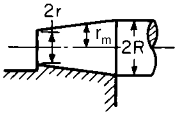


Fig. 3.2.13

Values of Coefficient of Friction For very low velocities of rotation (e.g., below 10 r/min), high loads, and with good lubrication, the coefficient of friction approaches the value of greasy friction, 0.07 to 0.15 (see Table 3.2.4). This is also the ''pullout'' coefficient of friction *on starting* the journal. With higher velocities, a fluid film is established between the journal and bearing, and the values of the coefficient of friction depend on the speed of rotation, the pressure on the bearing, and the viscosity of the oil. For journals running in complete bearing bushings, with a small clearance, i.e., with the diameter of the bushing slightly larger than the diameter of the journal, the experimental data of McKee give approximate values of the coefficient of friction as in Fig. 3.2.14.

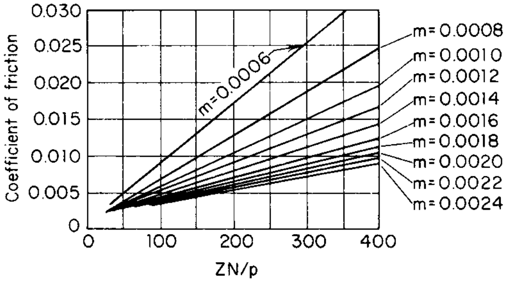


Fig. 3.2.14 Coefficient of friction of journal.

If d_1 is the diameter of the bushing in inches, d the diameter of the journal in inches, then $(d_1 - d)$ is the diametral clearance and $m = (d_1 - d)/d$ is the clearance ratio. The diagram of McKee (Fig. 3.2.14) gives the coefficient of friction as a function of the characteristic num-

Table 3.2.9 Coefficients of Friction for Worm Gears

Rubbing speed of worm, ft/min (m/min)	100 (30.5)	200 (61)	300 (91.5)	500 (152)	800 (244)	1200 (366)
Phosphor-bronze wheel, polished-steel worm	0.054	0.045	0.039	0.030	0.024	0.020
Single-threaded cast-iron worm and gear	0.060	0.051	0.047	0.034	0.025	

ber ZN/p, where N is the speed of rotation in revolutions per minute, $p = P/(dl)$ is the average pressure in lb/in^2 on the projected area of the bearing, P is the load, l is the axial length of the bearing, and Z is the absolute viscosity of the oil in centipoises. Approximate values of Z at 100 (130)°F are as follows: light machine oil, 30 (16); medium machine oil, 60 (25); medium-heavy machine oil, 120 (40); heavy machine oil, 160 (60).

For purposes of design of ordinary machinery with bearing pressures from 50 to 300 lb/in^2 (344.7 to 2,068 kN/m^2) and speeds of 100 to 3,000 rpm, values for the coefficient of journal friction can be taken from 0.008 to 0.020.

Thrust Bearings

Frictional Resistance for Flat Ring Bearing Step bearings or pivots may be used to resist the end thrust of shafts. Let L = total load in the direction of the shaft axis and f = coefficient of sliding friction.

For a **ring-shaped flat step bearing** such as that shown in Fig. 3.2.15 (or a **collar bearing**), the **moment of thrust friction** $M = \frac{1}{3} fL(D^3 - d^3)/(D^2 - d^2)$. For a **flat circular step bearing,** $d = 0$, and $M = \frac{1}{3} fLD$.

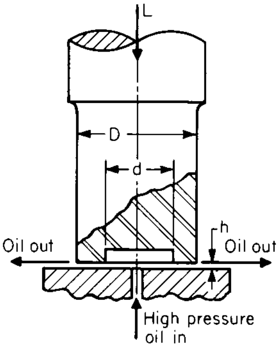


Fig. 3.2.15

The value of the coefficient of sliding friction is 0.08 to 0.15 when the speed of rotation is very slow. At higher velocities when a collar or step bearing is used, $f = 0.04$ to 0.06. If the design provides for the formation of a load carrying oil film, as in the case of the Kingsbury thrust bearing, the coefficient of friction has values $f = 0.001$ to 0.0025.

Where oil is supplied from an external pump with such pressure as to separate the surfaces and provide an oil film of thickness h (Fig. 3.2.15), the frictional moment is

$$M = \frac{Zn(D^4 - d^4)}{67 \times 10^7\, h} = \frac{\pi\mu\omega(D^4 - d^4)}{32\, h}$$

where D and d are in inches, μ is the absolute viscosity, ω is the angular velocity, h is the film thickness, in, Z is viscosity of lubricant in centipoises, and n is rotation speed, r/min. With this kind of lubrication the frictional moment depends upon the speed of rotation of the shaft and actually approaches zero for zero shaft speeds. The thrust load will be carried on a film of oil regardless of shaft rotation for as long as the pump continues to supply the required volume and pressure (see also Secs. 8 and 14).

Example. A hydrostatic thrust bearing carries 101,000 lb, D is 16 in, d is 10 in, oil-film thickness h is 0.006 in, oil viscosity Z, 30 centipoises at operating temperature, and n is 750 r/min. Substituting these values, the frictional torque M is 310 in · lb (358 cm · kg). The oil supply pressure was 82.5 lb/in^2 (569 kN/m^2); the oil flow, 12.2 gal/min (46.2 l/min).

Frictional Forces in Pin Joints of Mechanisms

In the absence of friction, or when the effect of friction is negligible, the force transmitted by the link b from the driver a to the driven link c (Figs. 3.2.16 and 3.2.17) acts through the centerline OO of the pins connecting the link b with links a and c. With friction, this line of action shifts to the line AA, tangent to small circles of diameter d. The diameter d of the circle, called the **friction circle,** for each individual joint, is equal to fD, where D is the diameter of the pin and f is the coefficient of friction between the pin and the link. The choice of the proper disposition of the tangent AA with respect to the two friction circles is dictated by the consideration that friction always opposes the action of the linkage. The force f opposes the motion of a; therefore, with friction it acts on a longer lever than without friction (Figs. 3.2.16 and 3.2.17). On the other hand, the force F drives the link c; friction hinders its action, and the equivalent lever is shorter with friction than without friction; the friction throws the line of action toward the center of rotation of link c.

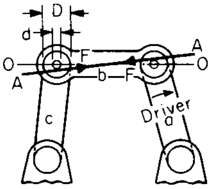


Fig. 3.2.16

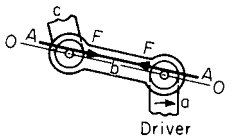


Fig. 3.2.17

Example. An engine eccentric (Fig. 3.2.18) is a joint where the friction loss may be large. For the dimensions shown and with a torque of 250 in · lb applied to the rotating shaft, the resultant horizontal force, with no friction, will act through the center of the eccentric and be 250/(2.5 sin 60) or 115.5 lb. With friction coefficient 0.1, the resultant force (which for a long rod remains approximately horizontal) will be tangent to the friction circle of radius 0.1 × 5, or 0.5 in, and have a magnitude of 250/(2.5 sin 60 + 0.5), or 93.8 lb (42.6 kg).

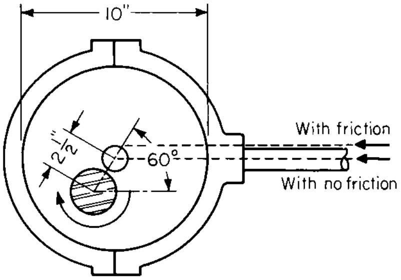


Fig. 3.2.18

Tension Elements

Frictional Resistance In Fig. 3.2.19, let T_1 and T_2 be the tensions with which a rope, belt, chain, or brake band is strained over a drum, pulley, or sheave, and let the rope or belt be on the point of slipping from T_2 toward T_1 by reason of the difference of tension $T_1 - T_2$. Then $T_1 - T_2$ = circumferential force P transferred by friction must be equal to the frictional resistance W of the belt, rope, or band on the drum or pulley. Also, let a = angle subtending the arc of contact between the drum and tension element. Then, disregarding centrifugal forces,

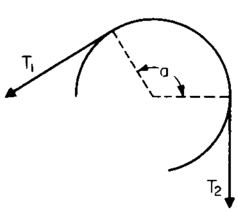


Fig. 3.2.19

$$T_1 = T_2 e^{fa} \text{ and } P = (e^{fa} - 1)T_1/e^{fa} = (e^{fa} - 1)T_2 = W$$

where e = base of the napierian system of logarithms = 2.178+.

Table 3.2.10 Values of e^{fa}

$\frac{a^\circ}{360^\circ}$	f								
	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0.1	1.06	1.1	1.13	1.17	1.21	1.25	1.29	1.33	1.37
0.2	1.13	1.21	1.29	1.37	1.46	1.55	1.65	1.76	1.87
0.3	1.21	1.32	1.45	1.60	1.76	1.93	2.13	2.34	2.57
0.4	1.29	1.46	1.65	1.87	2.12	2.41	2.73	3.10	3.51
0.425	1.31	1.49	1.70	1.95	2.23	2.55	2.91	3.33	3.80
0.45	1.33	1.53	1.76	2.03	2.34	2.69	3.10	3.57	4.11
0.475	1.35	1.56	1.82	2.11	2.45	2.84	3.30	3.83	4.45
0.5	1.37	1.60	1.87	2.19	2.57	3.00	3.51	4.11	4.81
0.525	1.39	1.64	1.93	2.28	2.69	3.17	3.74	4.41	5.20
0.55	1.41	1.68	2.00	2.37	2.82	3.35	3.98	4.74	5.63
0.6	1.46	1.76	2.13	2.57	3.10	3.74	4.52	5.45	6.59
0.7	1.55	1.93	2.41	3.00	3.74	4.66	5.81	7.24	9.02
0.8	1.65	2.13	2.73	3.51	4.52	5.81	7.47	9.60	12.35
0.9	1.76	2.34	3.10	4.11	5.45	7.24	9.60	12.74	16.90
1.0	1.87	2.57	3.51	4.81	6.59	9.02	12.35	16.90	23.14
1.5	2.57	4.11	6.59	10.55	16.90	27.08	43.38	69.49	111.32
2.0	3.51	6.59	12.35	23.14	43.38	81.31	152.40	285.68	535.49
2.5	4.81	10.55	23.14	50.75	111.32	244.15	535.49	1,174.5	2,575.9
3.0	6.59	16.90	43.38	111.32	285.68	733.14	1,881.5	4,828.5	12,391
3.5	9.02	27.08	81.31	244.15	733.14	2,199.90	6,610.7	19,851	59,608
4.0	12.35	43.38	152.40	535.49	1,881.5	6,610.7	23,227	81,610	286,744

NOTE: $e^{\pi} = 23.1407$, $\log e^{\pi} = 1.3643764$.

f is the static coefficient of friction (f_0) when there is no slip of the belt or band on the drum and the coefficient of kinetic friction (f) when slip takes place. For ease of computation, the values of the quantity e^{fa} are tabulated on Table 3.2.10.

Average values of f_0 for belts, ropes, and brake bands are as follows: for leather belt on cast-iron pulley, very greasy, 0.12; slightly greasy, 0.28; moist, 0.38. For hemp rope on cast-iron drum, 0.25; on wooden drum, 0.40; on rough wood, 0.50; on polished wood, 0.33. For iron brake bands on cast-iron pulleys, 0.18. For wire ropes, Tichvinsky reports coefficients of static friction, f_0, for a ⅝ rope (8 × 19) on a worn-in cast-iron groove: 0.113 (dry); for mylar on aluminum, 0.4 to 0.7.

Belt Transmissions; Effects of Belt Compliance

In the configuration of Fig. 3.2.20, pulley A drives a belt at angular velocity ω_A. Pulley B, here assumed to be of the same radius R as A, is driven at angular velocity ω_B. If the belt is extensible and the resistive torque $M = (T_1 - T_2)\,R$ is applied at B, ω_B will be smaller than ω_A and power will be dissipated at a rate $W = M(\omega_A - \omega_B)$. Likewise, the surface velocity V_1 of the more stretched belt will be larger than V_2. No slip will take place over the wraps A_T-A_S and B_T-B_S. The slip angles a_A and a_B can be calculated from

$$a_A = [\ln(T_1/T_2)]/f_A \qquad a_B = [\ln(T_1/T_2)]/f_B$$

where f_A and f_B are the coefficients of friction on pulleys A and B, respectively. To calculate the above values, it is necessary to know the mean tension of the belt, $T = (T_1 + T_2)/2$. Then, $T_1/T_2 = [T + M/(2R)]/[T - M/(2R)]$. In this configuration, when the slip angles become equal to π (180°), complete slip occurs.

It is interesting to note that torque is transmitted only over the slip arcs a_A and a_B since there is no tension variation in the arcs A_T-A_S and B_T-B_S where the belt is in a uniform state of stretch.

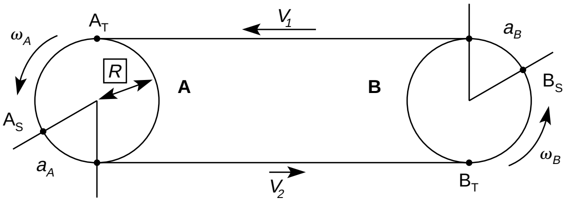


Fig. 3.2.20 Pulley transmission with extensible belt.

3.3 MECHANICS OF FLUIDS

by J. W. Murdock

REFERENCES: *Specific.* ‘‘Handbook of Chemistry and Physics,’’ Chemical Rubber Company. ‘‘Smithsonian Physical Tables,’’ Smithsonian Institution. ‘‘Petroleum Measurement Tables,’’ ASTM. ‘‘Steam Tables,’’ ASME. ‘‘American Institute of Physics Handbook,’’ McGraw-Hill. ‘‘International Critical Tables,’’ McGraw-Hill. ‘‘Tables of Thermal Properties of Gases,’’ *NBS Circular 564.* Murdock, ‘‘Fluid Mechanics and its Applications,’’ Houghton Mifflin, 1976. ‘‘Pipe Friction Manual,’’ Hydraulic Institute. ‘‘Flow of Fluids,’’ ASME, 1971. ‘‘Fluid Meters,’’ 6th ed., ASME, 1971. ‘‘Measurement of Fluid Flow in Pipes Using Orifice, Nozzle, and Venturi,’’ ASME Standard MFC-3M-1984. Murdock, ASME 64-WA/FM-6. Horton, *Engineering News,* 75, 373, 1916. Belvins, ASME 72/WA/FE-39. Staley and Graven, ASME 72PET/30. ‘‘Temperature Measurement,’’ PTC 19.3, ASME. Moody, *Trans. ASME,* 1944, pp. 671–684. *General.* Binder, ‘‘Fluid Mechanics,’’ Prentice-Hall. Langhaar, ‘‘Dimensional Analysis and Theory of Models,’’ Wiley. Murdock, ‘‘Fluid Mechanics,’’ Drexel University Press. Rouse, ‘‘Elementary Mechanics of Fluids,’’ Wiley. Shames, ‘‘Mechanics of Fluids,’’ McGraw-Hill. Streeter, ‘‘Fluid Mechanics,’’ McGraw-Hill.

Notation

a = acceleration, area, exponent
A = area
c = velocity of sound
C = coefficient
$\mathbf{C}$ = Cauchy number
$\mathbf{C}_p$ = pressure coefficient
d = diameter, distance
E = bulk modulus of elasticity, modulus of elasticity (Young's modulus), velocity of approach factor, specific energy
$\mathbf{E}$ = Euler number
f = frequency, friction factor
F = dimension of force, force
$\mathbf{F}$ = Froude number
g = acceleration due to gravity
g_c = proportionality constant = 32.1740 lmb/(lbf) (ft/s^2)
G = mass velocity
h = head, vertical distance below a liquid surface
H = geopotential altitude
i = ideal
I = moment of inertia
J = mechanical equivalent of heat, 778.169 ft · lbf
k = isentropic exponent, ratio of specific heats
K = constant, resistant coefficient, weir coefficient
$\mathbf{K}$ = flow coefficient
L = dimension of length, length
m = mass, lbm
$\dot{m}$ = mass rate of flow, lbm/s
M = dimension of mass, mass (slugs)
$\dot{M}$ = mass rate of flow, slugs/s
$\mathbf{M}$ = Mach number
n = exponent for a polytropic process, roughness factor
N = dimensionless number
p = pressure
P = perimeter, power
q = heat added
$\mathbf{q}$ = flow rate per unit width
Q = volumetric flow rate
r = pressure ratio, radius
R = gas constant, reactive force
$\mathbf{R}$ = Reynolds number
R_h = hydraulic radius
s = distance, second
sp. gr. = specific gravity
S = scale reading, slope of a channel
$\mathbf{S}$ = Strouhal number
t = time
T = dimension of time, absolute temperature
u = internal energy
U = stream-tube velocity
v = specific volume
V = one-dimensional velocity, volume
$\mathbf{V}$ = velocity ratio
W = work done by fluid
$\mathbf{W}$ = Weber number
x = abscissa
y = ordinate
Y = expansion factor
z = height above a datum
Z = compressibility factor, crest height
α = angle, kinetic energy correction factor
β = ratio of primary element diameter to pipe diameter
γ = specific weight
δ = boundary-layer thickness
ε = absolute surface roughness
θ = angle
μ = dynamic viscosity
ν = kinematic viscosity
π = 3.14159 . . . , dimensionless ratio
ρ = density
σ = surface tension
τ = unit shear stress
ω = rotational speed

FLUIDS AND OTHER SUBSTANCES

Substances may be classified by their response when at rest to the imposition of a shear force. Consider the two very large plates, one moving, the other stationary, separated by a small distance y as shown in Fig. 3.3.1. The space between these plates is filled with a substance whose surfaces adhere to these plates in such a manner that its upper surface moves at the same velocity as the upper plate and the lower surface is stationary. The upper surface of the substance attains a velocity of U as the result of the application of shear force F_s. As y approaches dy, U approaches dU, and the rate of deformation of the substance becomes dU/dy. The **unit shear stress** is defined by $\tau = F_s/A_s$, where A_s is the shear or surface area. The deformation characteristics of various substances are shown in Fig. 3.3.2.

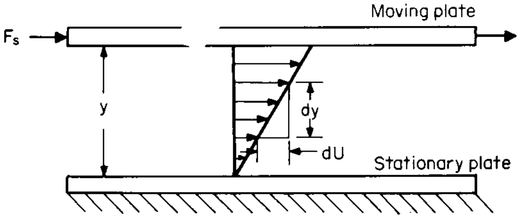


Fig. 3.3.1 Flow of a substance between parallel plates.

An ideal or **elastic solid** will resist the shear force, and its rate of deformation will be zero regardless of loading and hence is coincident with the ordinate of Fig. 3.3.2. A **plastic** will resist the shear until its yield stress is attained, and the application of additional loading will cause it to deform continuously, or flow. If the deformation rate is directly proportional to the flow, it is called an **ideal plastic**.

1

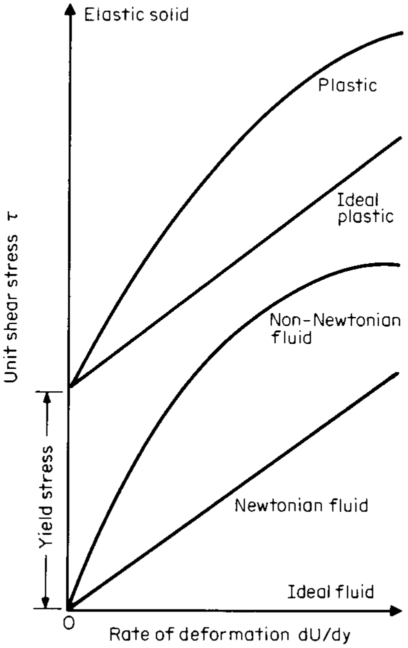


Fig. 3.3.2 Deformation characteristics of substances.

If the **substance** is unable to resist even the slightest amount of shear without flowing, it is a **fluid**. An **ideal fluid** has no internal friction, and hence its deformation rate coincides with the abscissa of Fig. 3.3.2. All real fluids have internal friction so that their rate of deformation is proportional to the applied shear stress. If it is directly proportional, it is called a **Newtonian fluid;** if not, a **non-Newtonian fluid**.

Two kinds of **fluids** are considered in this section, incompressible and compressible. A **liquid** except at very high pressures and/or temperatures may be considered incompressible. **Gases** and **vapors** are compressible fluids, but only **ideal gases** (those that follow the ideal-gas laws) are considered in this section. All others are covered in Secs. 4.1 and 4.2.

FLUID PROPERTIES

The **density** ρ of a fluid is its mass per unit volume. Its dimensions are M/L^3. In **fluid mechanics,** the units are slugs/ft^3 and lbf · s^2/ft^4) (515.3788 kg/m^3), but in **thermodynamics** (Sec. 4.1), the units are lbm/ft^3 (16.01846 kg/m^3). Numerical values of densities for selected liquids are shown in Table 3.3.1. The temperature change at 68°F (20°C) required to produce a 1 percent change in density varies from 12°F (6.7°C) for kerosene to 99°F (55°C) for mercury.

The **specific volume** v of a fluid is its volume per unit mass. Its dimensions are L^3/M. The units are ft^3/lbm. Specific volume is related to density by $v = 1/\rho g_c$, where g_c is the proportionality constant [32.1740 (lbm/lbf)(ft/s^2)]. **Specific volumes** of ideal gases may be computed from the equation of state: $v = RT/p$, where R is the gas constant in ft · lbf/(lbm)(°R) (see Sec. 4.1), T is the temperature in degrees Rankine (°F + 459.67), and p is the pressure in lbf/ft^2 abs.

The **specific weight** γ of a fluid is its weight per unit volume and has dimensions of F/L^3 or $M/(L^2)(T^2)$. The units are lbf/ft^3 or slugs/(ft^2)(s^2) (157.087 N/m^3). Specific weight is related to density by $\gamma = \rho g$, where g is the acceleration of gravity.

The **specific gravity** (sp. gr.) of a substance is a dimensionless ratio of the density of a fluid to that of a reference fluid. Water is used as the reference fluid for solids and liquids, and air is used for gases. Since the density of liquids changes with temperature for a precise definition of **specific gravity,** the temperature of the fluid and the reference fluid should be stated, for example, 60/60°F, where the upper temperature pertains to the liquid and the lower to water. If no temperatures are stated, reference is made to water at its maximum density, which occurs at 3.98°C and atmospheric pressure. The maximum density of water is 1.9403 slugs/ft^3 (999.973 kg/m^3). See Sec. 1.2 for conversion factors for API and Baumé hydrometers. For gases, it is common practice to use the ratio of the molecular weight of the gas to that of air (28.9644), thus eliminating the necessity of stating the pressure and temperature for ideal gases.

The **bulk modulus of elasticity** E of a fluid is the ratio of the pressure stress to the volumetric strain. Its dimensions are F/L^2. The units are lbf/in^2 or lbf/ft^2. E depends upon the thermodynamic process causing the change of state so that $E_x = -v(\partial p/\partial v)_x$, where x is the process. For ideal gases, $E_T = p$ for an isothermal process and $E_s = kp$ for an isentropic process where k is the ratio of specific heats. Values of E_T and E_S for liquids are given in Table 3.3.2. For liquids, a mean value is used by integrating the equation over a finite interval, or $E_{xm} = -v_1(\Delta p/\Delta v)_x = v_1(p_2 - p_1)/(v_1 - v_2)_x$.

EXAMPLE. What pressure must be applied to ethyl alcohol at 68°F (20°C) to produce a 1 percent decrease in volume at constant temperature?

$$\begin{aligned}\Delta p &= -E_T(\Delta v/v) = -(130{,}000)(-0.01)\\ &= 1{,}300\ \text{lbf/in}^2\ (9 \times 10^6\ \text{N/m}^2)\end{aligned}$$

In a like manner, the pressure required to produce a 1 percent decrease in the volume of mercury is found to be 35,900 lbf/in^2 (248 × 10^6 N/m^2). For most engineering purposes, liquids may be considered as incompressible fluids.

The **acoustic velocity,** or velocity of sound in a fluid, is given by $c = \sqrt{E_s/\rho}$. For an ideal gas $c = \sqrt{kp/\rho} = \sqrt{kg_c pv} = \sqrt{kg_c RT}$. Values of the speed of sound in liquids are given in Table 3.3.2.

EXAMPLE. Check the value of the velocity of sound in benzene at 68°F (20°C) given in Table 3.3.2 using the isentropic bulk modulus. $c = \sqrt{E_s/\rho} = \sqrt{144 \times 223{,}000/1.705} = 4{,}340$ ft/s (1,320 m/s). Additional information on the velocity of sound is given in Secs. 4, 11, and 12.

Application of shear stress to a fluid results in the continual and permanent distortion known as flow. **Viscosity** is the resistance of a fluid to shear motion—its internal friction. This resistance is due to two phenomena: (1) cohesion of the molecules and (2) molecular transfer from one layer to another, setting up a tangential or shear stress. In liquids, cohesion predominates, and since cohesion decreases with increasing temperature, the viscosity of liquids does likewise. Cohesion is relatively weak in gases; hence increased molecular activity with increasing temperature causes an increase in molecular transfer with corresponding increase in viscosity.

The **dynamic viscosity** μ of a fluid is the ratio of the shearing stress to the rate of deformation. From Fig. 3.3.1, $\mu = \tau/(dU/dy)$. Its dimensions are $(F)(T)/L^2$ or $M/(L)(T)$. The units are lbf · s/ft^2 or slugs/(ft)(s) [47.88026(N · s)/m^2].

In the cgs system, the unit of dynamic viscosity is the **poise,** 2,089 × 10^{-6} (lbf · s)/ft^2 [0.1 (N · s)/m^2], but for convenience the **centipoise** (1/100 poise) is widely used. The dynamic viscosity of water at 68°F (20°C) is approximately 1 centipoise.

Table 3.3.3 gives values of dynamic viscosity for selected liquids at atmospheric pressure. Values of viscosity for fuels and lubricants are given in Sec. 6. The effect of pressure on liquid viscosity is generally

Table 3.3.1 Density of Liquids at Atmospheric Pressure

Temp: °C	0	20	40	60	80	100
°F	32	68	104	140	176	212
Liquid	ρ, slugs/ft^3 (515.4 kg/m^3)					
Alcohol, ethyl[f]	1.564	1.532	1.498	1.463		
Benzene[a,b]	1.746	1.705	1.663	1.621	1.579	
Carbon tetrachloride[a,b]	3.168	3.093	3.017	2.940	2.857	
Gasoline,[c] sp. gr. 0.68	1.345	1.310	1.275	1.239		
Glycerin[a,b]	2.472	2.447	2.423	2.398	2.372	2.346
Kerosene,[c] sp. gr. 0.81	1.630	1.564	1.536	1.508	1.480	
Mercury[b]	26.379	26.283	26.188	26.094	26.000	25.906
Oil, machine,[c] sp. gr. 0.907	1.778	1.752	1.727	1.702	1.677	1.651
Water, fresh[d]	1.940	1.937	1.925	1.908	1.885	1.859
Water, salt[e]	1.995	1.988	1.975			

SOURCES: Computed from data given in:

[a] ''Handbook of Chemistry and Physics,'' 52d ed., Chemical Rubber Company, 1971–1972.

[b] ''Smithsonian Physical Tables,'' 9th rev. ed., 1954.

[c] ASTM-IP, ''Petroleum Measurement Tables.''

[d] ''Steam Tables,'' ASME, 1967.

[e] ''American Institute of Physics Handbook,'' 3d ed., McGraw-Hill, 1972.

[f] ''International Critical Tables,'' McGraw-Hill.

Table 3.3.2 Bulk Modulus of Elasticity, Ratio of Specific Heats of Liquids and Velocity of Sound at One Atmosphere and 68°F (20°C)

Liquid	E in lbf/in^2 (6,895 N/m^2) Isothermal E_T	Isentropic E_s	$k = c_p/c_v$	c in ft/s (0.3048 m/s)
Alcohol, ethyl[a,e]	130,000	155,000	1.19	3,810
Benzene[a,f]	154,000	223,000	1.45	4,340
Carbon tetrachloride[a,b]	139,000	204,000	1.47	3,080
Glycerin[f]	654,000	719,000	1.10	6,510
Kerosene,[a,e] sp. gr. 0.81	188,000	209,000	1.11	4,390
Mercury[e]	3,590,000	4,150,000	1.16	4,770
Oil, machine,[f] sp. gr. 0.907	189,000	219,000	1.13	4,240
Water, fresh[a]	316,000	319,000	1.01	4,860
Water, salt[a,e]	339,000	344,000	1.01	4,990

SOURCES: Computed from data given in:
[a] ''Handbook of Chemistry and Physics,'' 52d ed., Chemical Rubber Company, 1971–1972.
[b] ''Smithsonian Physical Tables,'' 9th rev. ed., 1954.
[c] ASTM-IP, ''Petroleum Measurement Tables.''
[d] ''Steam Tables,'' ASME, 1967.
[e] ''American Institute of Physics Handbook,'' 3d ed., McGraw-Hill, 1972.
[f] ''International Critical Tables,'' McGraw-Hill.

Table 3.3.3 Dynamic Viscosity of Liquids at Atmospheric Pressure

Temp:						
°C	0	20	40	60	80	100
°F	32	68	104	140	176	212
Liquid	μ, (lbf · s)/(ft^2) [47.88 (N · s)/(m^2)] × 10^6					
Alcohol, ethyl[a,e]	37.02	25.06	17.42	12.36	9.028	
Benzene[a]	19.05	13.62	10.51	8.187	6.871	
Carbon tetrachloride[e]	28.12	20.28	15.41	12.17	9.884	
Gasoline,[b] sp. gr. 0.68	7.28	5.98	4.93	4.28		
Glycerin[d]	252,000	29,500	5,931	1,695	666.2	309.1
Kerosene,[b] sp. gr. 0.81	61.8	38.1	26.8	20.3	16.3	
Mercury[a]	35.19	32.46	30.28	28.55	27.11	25.90
Oil, machine,[a] sp. gr. 0.907						
''Light''	7,380	1.810	647	299	164	102
''Heavy''	66,100	9,470	2,320	812	371	200
Water, fresh[c]	36.61	20.92	13.61	9.672	7.331	5.827
Water, salt[d]	39.40	22.61	18.20			

SOURCES: Computed from data given in:
[a] ''Handbook of Chemistry and Physics,'' 52d ed., Chemical Rubber Company, 1971–1972.
[b] ''Smithsonian Physical Tables,'' 9th rev. ed., 1954.
[c] ''Steam Tables,'' ASME, 1967.
[d] ''American Institute of Physics Handbook,'' 3d ed., McGraw-Hill, 1972.
[e] ''International Critical Tables,'' McGraw-Hill.

Table 3.3.4 Viscosity of Gases at One Atmosphere

Temp:									
°C	0	20	60	100	200	400	600	800	1000
°F	32	68	140	212	392	752	1112	1472	1832
Gas	μ, (lbf · s)/(ft^2) [47.88(N · s)/(m^2)] × 10^8								
Air*	35.67	39.16	41.79	45.95	53.15	70.42	80.72	91.75	100.8
Carbon dioxide*	29.03	30.91	35.00	38.99	47.77	62.92	74.96	87.56	97.71
Carbon monoxide†	34.60	36.97	41.57	45.96	52.39	66.92	79.68	91.49	102.2
Helium*	38.85	40.54	44.23	47.64	55.80	71.27	84.97	97.43	
Hydrogen*,†	17.43	18.27	20.95	21.57	25.29	32.02	38.17	43.92	49.20
Methane*	21.42	22.70	26.50	27.80	33.49	43.21			
Nitrogen*,†	34.67	36.51	40.14	43.55	51.47	65.02	76.47	86.38	95.40
Oxygen†	40.08	42.33	46.66	50.74	60.16	76.60	90.87	104.3	116.7
Steam‡		18.49	21.89	25.29	33.79	50.79	67.79	84.79	

SOURCES: Computed from data given in:
* ''Handbook of Chemistry and Physics,'' 52d ed., Chemical Rubber Company, 1971–1972.
† ''Tables of Thermal Properties of Gases,'' *NBS Circular* 564, 1955.
‡ ''Steam Tables,'' ASME, 1967.

unimportant in fluid mechanics except in lubricants (Sec. 6). The viscosity of water changes little at pressures up to 15,000 lbf/in^2, but for animal and vegetable oils it increases about 350 percent and for mineral oils about 1,600 percent at 15,000 lbf/in^2 pressure.

The dynamic viscosity of gases is primarily a temperature function and essentially independent of pressure. Table 3.3.4 gives values of dynamic viscosity of selected gases.

The **kinematic viscosity** ν of a fluid is its dynamic viscosity divided by its density, or $\nu = \mu/\rho$. Its dimensions are L^2/T. The units are ft^2/s (9.290304×10^{-2} m^2/s).

In the cgs system, the unit of kinematic viscosity is the **stoke** (1×10^{-4} m^2/s^2), but for convenience, the **centistoke** (1/100 stoke) is widely used. The kinematic viscosity of water at 68°F (20°C) is approximately 1 centistoke.

The standard device for **experimental determination of kinematic viscosity** in the United States is the **Saybolt Universal** viscometer. It consists essentially of a metal tube and an orifice built to rigid specifications and calibrated. The time required for a gravity flow of 60 cubic centimeters is called the SSU (Saybolt seconds Universal). Approximate conversions of SSU to stokes may be made as follows:

$$32 < \text{SSU} < 100 \text{ seconds, stokes} = 0.00226\,(\text{SSU}) - 1.95/(\text{SSU})$$
$$\text{SSU} > 100 \text{ seconds, stokes} = 0.00220\,(\text{SSU}) - 1.35/(\text{SSU})$$

For viscous oils, the **Saybolt Furol** viscometer is used. Approximate conversions of SSF (saybolt seconds Furol) may be made as follows:

$$25 < \text{SSF} < 40 \text{ seconds, stokes} = 0.0224\,(\text{SSF}) - 1.84/(\text{SSF})$$
$$\text{SSF} > 40 \text{ seconds, stokes} = 0.0216\,(\text{SSF}) - 0.60/(\text{SSF})$$

For exact conversions of Saybolt viscosities, see ASTM D445-71 and Sec. 6.11.

The **surface tension** σ of a fluid is the work done in extending the surface of a liquid one unit of area or work per unit area. Its dimensions are F/L. The units are lbf/ft (14.5930 N/m).

Values of σ for various interfaces are given in Table 3.3.5. Surface tension decreases with increasing temperature. Surface tension is of importance in the formation of bubbles and in problems involving atomization.

Table 3.3.5 Surface Tension of Liquids at One Atmosphere and 68°F (20°C)

	δ, lbf/ft (14.59 N/m) $\times 10^3$		
Liquid	In vapor	In air	In water
Alcohol, ethyl*	1.56	1.53	
Benzene*	2.00	1.98	2.40
Carbon tetrachloride*	1.85	1.83	3.08
Gasoline,* sp. gr. 0.68	1.3–1.6		2.7–3.6
Glycerin*	4.30	4.35	
Kerosene,* sp. gr. 0.81	1.6–2.2		
Mercury*	32.6§	32.8	25.7
Oil, machine,‡ sp. gr. 0.907	2.5	2.6	2.3–3.7
Water, fresh‡		4.99	
Water, salt‡		5.04	

SOURCES: Computed from data given in:
* ''International Critical Tables,'' McGraw-Hill.
† ASTM-IP, ''Petroleum Measurement Tables.''
‡ ''American Institute of Physics Handbook,'' 3d ed., McGraw-Hill, 1972.
§ In vacuum.

Capillary action is due to surface tension, **cohesion** of the liquid molecules, and the **adhesion** of the molecules on the surface of a solid. This action is of importance in fluid mechanics because of the formation of a meniscus (curved section) in a tube. When the adhesion is greater than the cohesion, a liquid ''wets'' the solid surface, and the liquid will rise in the tube and conversely will fall if the reverse. Figure 3.3.3 illustrates this effect on manometer tubes. In the reading of a manometer, all data should be taken at the center of the meniscus.

The **vapor pressure** pv of a fluid is the pressure at which its liquid and vapor are in equilibrium at a given temperature. See Secs. 4.1 and 4.2 for further definitions and values.

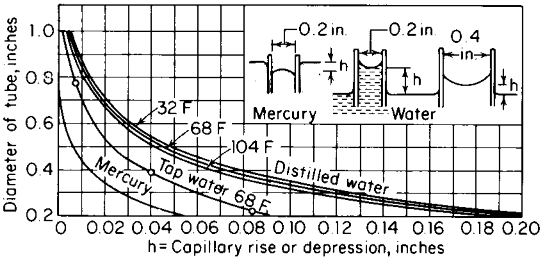


Fig. 3.3.3 Capillarity in circular glass tubes.

FLUID STATICS

Pressure p is the force per unit area exerted on or by a fluid and has dimensions of F/L^2. In fluid mechanics and in thermodynamic equations, the units are lbf/ft^2 (47.88026 N/m^2), but engineering practice is to use units of lbf/in^2 (6,894.757 N/m^2).

The relationship between **absolute pressure, gage pressure,** and **vacuum** is shown in Fig. 3.3.4. Most fluid-mechanics equations and **all** thermodynamic equations require the use of **absolute pressure,** and unless otherwise designated, a pressure should be understood to be **absolute pressure**. Common practice is to denote absolute pressure as lbf/ft^2 abs, or psfa, lbf/in^2 abs or psia; and in a like manner for **gage pressure** lbf/ft^2 g, lbf/in^2 g, and psig.

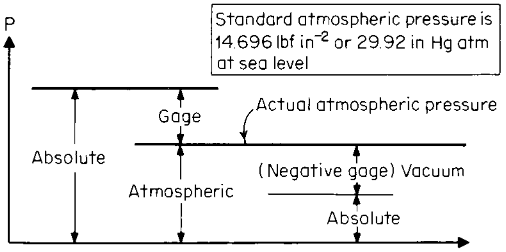


Fig. 3.3.4 Pressure relations.

According to **Pascal's** principle, the pressure in a static fluid is the same in all directions.

The **basic equation of fluid statics** is obtained by consideration of a fluid particle at rest with respect to other fluid particles, all being subjected to body-force accelerations of a_x, a_y, and a_z opposite the directions of x, y, and z, respectively, and the acceleration of gravity in the z direction, resulting in the following:

$$dp = -\rho[a_x\,dx + a_y\,dy + (a_z + g)\,dz]$$

Pressure-Height Relations For a fluid at rest and subject only to the gravitational force, a_x, a_y, and a_z are zero and the **basic equation for fluid statics reduces to** $dp = -\rho g\,dz = \gamma\,dz$.

Liquids (Incompressible Fluids) The pressure-height equation integrates to $(p_1 - p_2) = \rho g(z_2 - z_1) = \gamma(z_2 - z_1) = \Delta p = \gamma h$, where h is measured from the liquid surface (Fig. 3.3.5).

EXAMPLE. A large closed tank is partly filled with 68°F (20°C) benzene. If the pressure on the surface is 10 lb/in^2, what is the pressure in the benzene at a depth of 11 ft below the liquid surface?

$$p_1 = \rho gh + p_2 = \frac{1.705 \times 32.17 \times 11}{144} + 10$$
$$= 14.19 \text{ lbf/in}^2\ (9.784 \times 10^4 \text{ N/m}^2)$$

Ideal Gases (Compressible Fluids) For problems involving the upper atmosphere, it is necessary to take into account the variation of gravity with altitude. For this purpose, the **geopotential altitude** H is used, defined by $H = Z/(1 + z/r)$, where r is the radius of the earth ($\approx 21 \times 10^6$ ft $\approx 6.4 \times 10^6$ m) and z is the height above sea level. The integration of the pressure-height equation depends upon the thermodynamic process. For an **isothermal process** $p_2/p_1 = e^{-(H_2 - H_1)/RT}$ and for a **polytropic process** ($n \neq 1$)

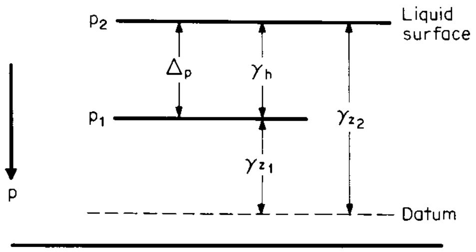


Fig. 3.3.5 Pressure equivalence.

$$\frac{p_2}{p_1} = \left[1 - \frac{(n-1)(H_2 - H_1)}{nRT_1}\right]^{n/(n-1)}$$

Temperature-height relations for a polytropic process ($n \neq 1$) are given by

$$\frac{n}{1-n} = \frac{H_2 - H_1}{R(T_2 - T_1)}$$

Substituting in the **pressure-altitude** equation,

$$p_2/p_1 = (T_2/T_1)^{(H_2 - H_1) \div R(T_1 - T_2)}$$

EXAMPLE. The U.S. Standard Atmosphere 1962 (Sec. 11) is defined as having a sea-level temperature of 59°F (15°C) and a pressure of 2,116.22 lbf/ft². From sea level to a geopotential altitude of 36,089 ft (11,000 m) the temperature decreases linearly with altitude to − 69.70°F (− 56.5°C). Check the value of pressure ratio at this altitude given in the standard table.

Noting that $T_1 = 59 + 459.67 = 518.67$, $T_2 = -69.70 + 459.67 = 389.97$, and $R = 53.34$ ft·lbf/(lbm)(°R),

$$\begin{aligned} p_2/p_1 &= (T_2/T_1)^{(H_2 - H_1)/R(T_1 - T_2)} \\ &= (389.97/518.67)^{(36{,}089\text{-}0)/53.34(518.67 - 389.97)} \\ &= 0.2233 \text{ vs. tabulated value of } 0.2234 \end{aligned}$$

Pressure-Sensing Devices The two principal devices using liquids are the **barometer** and the **manometer**. The barometer senses absolute pressure and the manometer senses pressure differential. For discussion of the barometer and other pressure-sensing devices, refer to Sec. 16.

Manometers are a direct application of the basic equation of fluid statics and serve as a pressure standard in the range of ⅒ in of water to 100 lbf/in². The most familiar type of manometer is the **U tube** shown in Fig. 3.3.6*a*. Because of the necessity of observing both legs simultaneously, the **well** or **cistern** type (Fig. 3.3.6*b*) is sometimes used. The **inclined manometer** (Fig. 3.3.6*c*) is a special form of the well-type manometer designed to enhance the readability of small pressure differentials. Application of the basic equation of fluid statics to each of the types results in the following equations. For the **U tube,** $p_1 - p_2 = (\gamma_m - \gamma_f)h$, where γ_m and γ_f are the specific weights of the manometer and sensed fluids, respectively, and h is the vertical distance between the liquid interfaces. For the **well type,** $p_1 - p_2 = (\gamma_m - \gamma_f)(z_2) \times (1 + A_2/A_1)$, where A_1 and A_2 are as shown in Fig. 3.3.6*b* and z_2 is the vertical distance from the fill line to the upper interface. Commercial manufacturers of well-type manometers correct for the area ratios so that $p_1 - p_2 = (\gamma_m - \gamma_f)S$, where S is the scale reading and is equal to $z_1(1 + A_2/A_1)$. For this reason, scales should not be interchanged between U type or well type or between well types without consulting the manufacturer. For inclined manometers,

$$p_1 - p_2 = (\gamma_m - \gamma_f)(A_2/A_1 + \sin\theta)R$$

where R is the distance along the inclined tube. Commercial inclined manometers also have special scales so that $p_1 - p_2 = (\gamma_m - \gamma_f)S$, where $S = (A_2/A_1 + \sin\theta)R$.

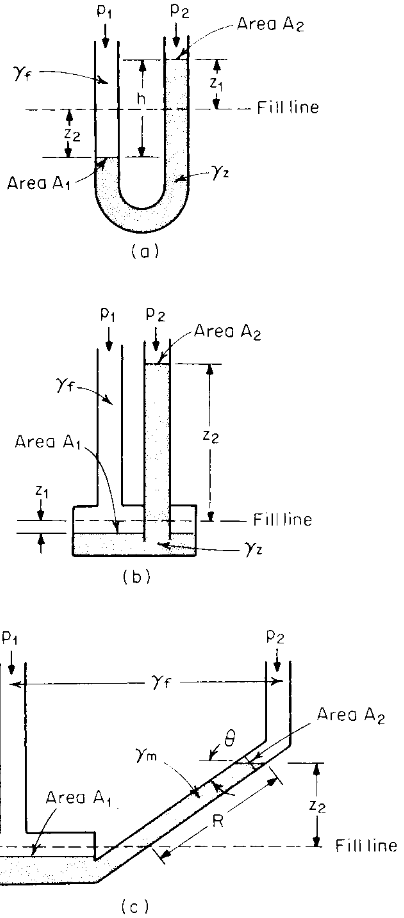


Fig. 3.3.6 *(a)* U-tube manometer; *(b)* well or cistern-type manometer; *(c)* inclined manometer.

EXAMPLE. A U-tube manometer containing mercury is used to sense the difference in water pressure. If the height between the interfaces is 10 in and the temperature is 68°F (20°C), what is the pressure differential?

$$\begin{aligned} p_1 - p_2 &= (\gamma_m - \gamma_f)h = g(\rho_m - \rho_f)h \\ &= 32.17(26.283 - 1.937)(10/12) \\ &= 652.7 \text{ lbf/ft}^2 \ (3.152 \times 10^4 \text{ N/m}^2) \end{aligned}$$

Liquid Forces The force exerted by a liquid on a **plane submerged surface** (Fig. 3.3.7) is given by $F = \int p\,dA = \gamma\int h \cdot dA = \gamma\,h, A$, where h_c is the distance from the liquid surface to the center of gravity of the surface, and A is the area of the surface. The location of the center of this force is given by

$$s_F = s_c + I_G/s_cA$$

where s_F is the inclined distance from the liquid surface to the center of force, s_c the inclined distance to the center of gravity of the surface, and I_G the moment of inertia around its center of gravity. Values of I_G are given in Sec. 5.2. See also Sec. 3.1. From Fig. 3.3.7, $h = R\sin\theta$, so that the vertical center of force becomes

$$h_F = h_c + I_G(\sin\theta)^2/h_cA$$

Example. Determine the force and its location acting on a rectangular gate 3 ft wide and 5 ft high at the bottom of a tank containing 68°F (20°C) water, 12 ft deep, (1) if the gate is vertical, and (2) if it is inclined 30° from horizontal.

1. *Vertical gate*

$F = \gamma h_c A = \rho g h_c A$
$= 1.937 \times 32.17(12 - 5/2)(5 \times 3)$
$= 8{,}800\ \text{lb}_f (3.914 \times 10^4\ \text{N})$
$h_F = h_c + I_G(\sin\theta)^2/h_c A$, from Sec. 5.2, I_G for a rectangle = (width)(height)^3/12
$h_F = (12 - 5/2) + (3 \times 5^3/12)(\sin 90°)^2/(12 - 5/2)(3 \times 5)$
$h_F = 9.719$ ft (2.962 m)

2. *Inclined gate*

$F = \gamma h_c A = \rho g h_c A$
$= 1.937 \times 32.17(12 - 5/2 \sin 30°)(5 \times 3)$
$= 10{,}048$ lbf $(4.470 \times 10^4$ N)
$h_F = h_c + I_G(\sin\theta)^2/h_c A$
$= (12 - 5/2 \sin 30°) + (3 \times 5^3/12)(\sin 30°)^2/(12 - 5/2 \sin 30°)(3 \times 5)$
$= 10.80$ ft (3.291 m)

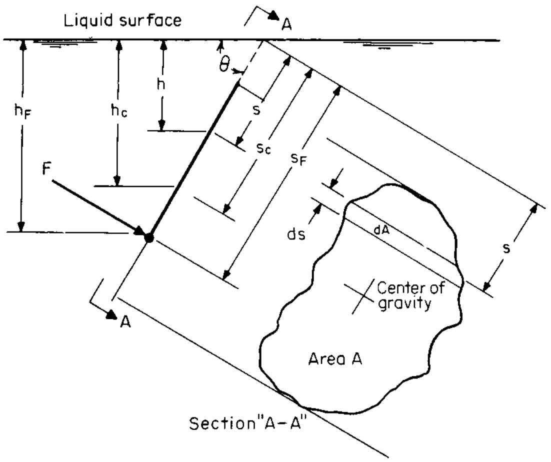


Fig. 3.3.7 Notation for liquid force on submerged surfaces.

Forces on irregular surfaces may be obtained by considering their horizontal and vertical components. The vertical component F_z equals the weight of liquid above the surface and acts through the centroid of the volume of the liquid above the surface. The horizontal component F_x equals the force on a vertical projection of the irregular surface. This force may be calculated by $F_x = \gamma h_{cx} A_x$, where h_{cx} is the distance from the surface center of gravity of the horizontal projection, and A_x is the projected area. The forces may be combined by $F = \sqrt{F_z^2 + F_z^2}$.

When fluid masses are **accelerated** without relative motion between fluid particles, the basic equation of fluid statics may be applied. For **translation** of a **liquid** mass due to uniform acceleration, the basic equation integrates to

$$p_2 - p_1 = -\rho[(x_2 - x_1)a_x + (y_2 - y_1)a_y + (z_2 - z_1)(a_z + g)]$$

Example. An open tank partly filled with a liquid is being accelerated up an inclined plane as shown in Fig. 3.3.8. The uniform acceleration is 20 ft/s^2 and the angle of the incline is 30°. What is the angle of the free surface of the liquid? Noting that on the free surface $p_2 = p_1$ and that the acceleration in the y direction is zero, the basic equation reduces to

$$(x_2 - x_1)a_x + (z_2 - z_1)(a_z + g) = 0$$

Solving for tan θ,

$$\tan\theta = \frac{z_1 - z_2}{x_2 - x_1} = \frac{a_x}{a_z + g} = \frac{a\cos\alpha}{a\sin\alpha + g}$$

$$= (20\cos 30°)/(20\sin 30° + 32.17) = 0.4107$$
$$\theta = 22°20'$$

For **rotation** of liquid masses with uniform rotational acceleration, the basic equation integrates to

$$p_2 - p_1 = \rho\left[\frac{\omega^2}{2}(x_2^2 - x_1^2) - g(z_2 - z_1)\right]$$

where ω is the rotational speed in rad/s and x is the radial distance from the axis of rotation.

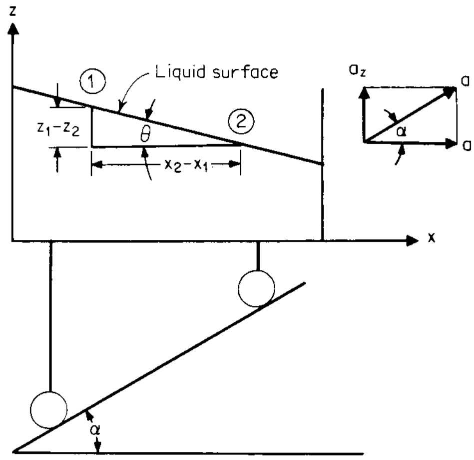


Fig. 3.3.8 Notation for translation example.

Example. The closed cylindrical tank shown in Fig. 3.3.9 is 4 ft in diameter and 10 ft high and is filled with 104°F (40°C) benzene. The tank is rotated at 250 r/min about an axis 3 ft from its centerline. Compute the maximum pressure differential in the tank. Analysis of the rotation equation indicates that the maximum pressure will occur at the maximum rotational radius and the minimum elevation and, conversely, the minimum at the minimum rotational radius and maximum elevation. From Fig. 3.3.9, $x_1 = 3 - 4/2 = 1$ ft, $x_2 = 1 + 4 = 5$ ft, $z_2 - z_1 = -10$ ft, and the rotational speed $\omega = 2\pi N/60 = 2\pi(250)/60 =$

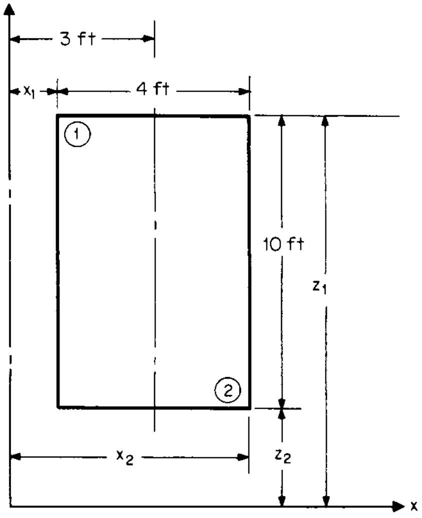


Fig. 3.3.9 Notation for rotation example.

26.18 rad/s. Substituting into the rotational equation,

$$p_2 - p_1 = \rho \left[\frac{\omega^2}{2} (x_2^2 - x_1^2) - g(z_2 - z_1) \right]$$
$$= \frac{1.663}{144} \left[\frac{(26.18)^2}{2} (5^2 - 1^2) - 32.17(-10) \right]$$
$$= 98.70 \text{ lbf/in}^2 \ (6.805 \times 10^5 \text{ N/m}^2)$$

Buoyancy Archimedes' principle states that a body immersed in a fluid is buoyed up by a force equal to the weight of the fluid displaced. If an object immersed in a fluid is heavier than the fluid displaced, it will sink to the bottom, and if lighter, it will rise. From the free-body diagram of Fig. 3.3.10, it is seen that for vertical equilibrium,

$$\Sigma F_z = 0 = F_B - F_g - F_D$$

where F_B is the buoyant force, F_g the gravity force (weight of body), and F_D the force required to prevent the body from rising. The buoyant force being the weight of the displaced liquid, the equilibrium equation may be written as

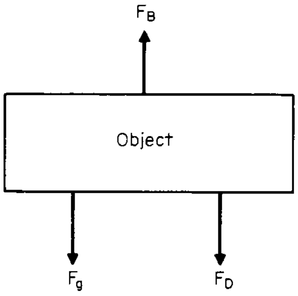


Fig. 3.3.10 Free body diagram of an immersed object.

$$F_D = F_B - F_g = \gamma_f V - \gamma_0 V = (\gamma_f - \gamma_0)V$$

where γ_f is the specific weight of the fluid, γ_0 is the specific weight of the object, and V is the volume of the object.

EXAMPLE. An airship has a volume of 3,700,000 ft³ and is filled with hydrogen. What is its gross lift in air at 59°F (15°C) and 14.696 psia? Noting that $\gamma = p/RT$,

$$F_D = (\gamma_f - \gamma_0)V = \left(\frac{p}{R_a T} - \frac{p}{R_{H_2} T} \right) V$$
$$= \frac{pV}{T} \left(\frac{1}{R_a} - \frac{1}{R_{H_2}} \right)$$
$$= \frac{144 \times 14.696 \times 3{,}700{,}000}{59 + 459.7} \left(\frac{1}{53.34} - \frac{1}{766.8} \right)$$
$$= 263{,}300 \text{ lbf } (1.171 \times 10^6 \text{ N})$$

Flotation is a special case of buoyancy where $F_D = 0$, and hence $F_B = F_g$.

EXAMPLE. A crude **hydrometer** consists of a cylinder of ½ in diameter and 2 in length surmounted by a cylinder of ⅛ in diameter and 10 in long. Lead shot is added to the hydrometer until its total weight is 0.32 oz. To what depth would this hydrometer float in 104°F (40°C) glycerin? For flotation, $F_B = F_g = \gamma_f V = \rho_f g V$ or $V = F_B/\rho_f g = (0.32/16)/(2.423 \times 32.17) = 2.566 \times 10^{-4}$ ft³. Volume of cylindrical portion of hydrometer $= V_c = \pi D^2 L/4 = \pi(0.5/12)^2(2/12)/4 = 2.273 \times 10^{-4}$ ft³. Volume of stem immersed $= V_S = V - V_C = 2.566 \times 10^{-4} - 2.273 \times 10^{-4} = 2.930 = 10^{-5}$ ft³. Length of immersed stem $= L_S = 4\, V_S/\pi D^2 = (4 \times 2.930 \times 10^{-5})/\pi(0.125/12)^2 = 0.3438$ ft $= 0.3438 \times 12 = 4.126$ in. Total immersion $= L + L_S = 2 + 4.126 = 6.126$ in (0.156 m).

Static Stability A body is in static equilibrium when the imposition of a small displacement brings into action forces that tend to restore the body to its original position. For **completely submerged bodies,** the center of buoyancy and the center of gravity must lie on the same vertical line and the center of buoyancy must be located above the center of gravity. Figure 3.3.11*a* shows a balloon and its basket in its normal position with the center of buoyancy *B* above and on the same vertical line as the center of gravity *G*. Figure 3.3.11*b* shows the balloon displaced from its normal position. In this position, there is a couple $F_g x$ which tends to restore the balloon and its basket to its original position. For **floating bodies,** the center of gravity and the center of buoyancy must lie on the same vertical line, but the center of buoyancy may be below the center of gravity, as is common practice in surface-ship design. It is required that when displaced, the line of action of the buoyant force intersect the centerline above the center of gravity. Figure 3.3.12*a* shows a floating body in its normal position with its center of gravity *G* on the same vertical line and above the center of buoyancy *B*. Figure 3.3.12*b* shows the object displaced. The intersection of the line of action of the buoyant force with the centerline of the body at *M* is called the metacenter. As shown, this above the center of buoyancy and sets up a restoring couple. When the metacenter is below the center of gravity, the object will capsize (see Sec. 11.3).

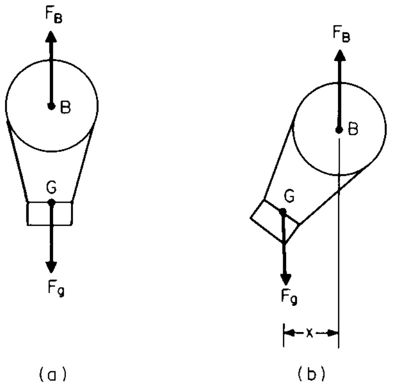


Fig. 3.3.11 Stability of an immersed body.

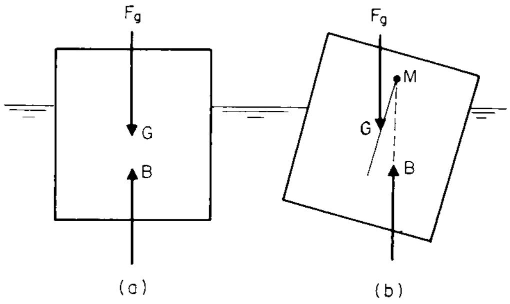


Fig. 3.3.12 Stability of a floating body.

FLUID KINEMATICS

Steady and Unsteady Flow If at every point in the fluid stream, none of the local fluid properties changes with time, the flow is said to be steady. The mathematical conditions for steady flow are met when $\partial(\text{fluid properties})/\partial t = 0$. While flow is generally unsteady by nature, many real cases of unsteady flow may be treated as steady flow by using average properties or by changing the space reference. The amount of error produced by the averaging technique depends upon the nature of the unsteady flow, but the latter technique is error-free when it can be applied.

Streamlines and Stream Tubes Velocity has both magnitude and direction and hence is a vector. A **streamline** is a line which gives the *direction* of the velocity of a fluid particle at each point in the flow stream. When streamlines are connected by a closed curve in steady flow, they will form a boundary through which the fluid particles cannot pass. The space between the streamlines becomes a **stream tube**. The stream-tube concept broadens the application of fluid-flow principles; for example, it allows treating the flow inside a pipe and the flow around an object with the same laws. A stream tube of decreasing size approaches its own axis, a central streamline; thus equations developed for a stream tube may also be applied to a streamline.

Velocity and Acceleration In the most general case of fluid motion, the resultant **velocity** U along a streamline is a function of both distance s and time t, or $U = f(s, t)$. In differential form,

$$dU = \frac{\partial U}{\partial s}\, ds + \frac{\partial U}{\partial t}\, dt$$

An expression for **acceleration** may be obtained by dividing the velocity equation by dt, resulting in

$$\frac{dU}{dt} = \frac{\partial U}{\partial s}\frac{ds}{dt} + \frac{\partial U}{\partial t}$$

for steady flow $\partial U/\partial t = 0$.

Velocity Profile In the flow of real fluids, the individual streamlines will have different velocities past a section. Figure 3.3.13 shows the steady flow of a fluid past a section (A-A) of a circular pipe. The **velocity profile** is obtained by plotting the velocity U of each streamline as it passes A-A. The stream tube that is formed by the space between the streamlines is the annulus whose area is dA, as shown in Fig. 3.3.13 for the stream tube whose velocity is U. The volumetric rate of flow Q for the flow past section A-A is $Q = \int U\, dA$. All flows take place between boundaries that are **three-dimensional**. The terms **one-dimensional, two-dimensional,** and **three-dimensional** flow refer to the number of dimensions required to describe the velocity profile. For **three-dimensional** flow, a **volume** (L^3) is required; for example, the flow of a fluid in a circular pipe. For **two-dimensional** flow, an **area** (L^2) is necessary; for example, the flow between two parallel plates. For **one-dimensional** flow, a **line** (L) describes the profile. In cases of two- or three-dimensional flow, $\int U\, dA$ can be integrated either mathematically if the equations are known or graphically if velocity-measurement data are available. In many engineering applications, the **average velocity** V may be used where $V = Q/A = (1/A)\int U\, dA$.

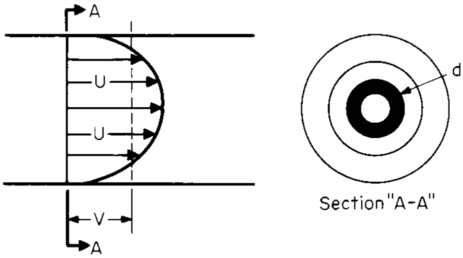


Fig. 3.3.13 Velocity profile.

The **continuity equation** is a special case of the general physical law of the conservation of mass. It may be simply stated for a control volume:

Mass rate entering = mass rate of storage + mass rate leaving

This may be expressed mathematically as

$$\rho U\, dA = \left[\frac{\partial}{\partial t}(\rho\, dA\, ds)\right] + \left[\rho U\, dA + \frac{\partial}{\partial s}(\rho U\, dA)\, ds\right]$$

where ds is an incremental distance along the control volume. For steady flow, $\partial/\partial t\,(\rho\, dA\, ds) = 0$, the general equation reduces to $d(\rho U\, dA) = 0$. Integrating the steady-flow **continuity equation** for the average velocity along a flow passage:

$$\rho VA = \text{a constant} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \ldots = \rho_n V_n A_n = \dot{M}$$

where $\dot{M}$ is the mass flow rate in slugs/s (14.5939 kg/s). In many engineering applications, the flow rate in pounds mass per second is desired, so that

$$\dot{m} = \frac{V_1 A_1}{v_1} = \frac{V_2 A_2}{v_2} = \ldots = \frac{V_n A_n}{v_n}$$

where $\dot{m}$ is the flow rate in lbm/s (0.4535924 kg/s).

EXAMPLE. Air discharges from a 12-in-diameter duct through a 4-in-diameter nozzle into the atmosphere. The pressure in the duct is 20 lbf/in^2, and atmospheric pressure is 14.7 lbf/in^2. The temperature of the air in the duct just upstream of the nozzle is 150°F, and the temperature in the jet is 147°F. If the velocity in the duct is 18 ft/s, compute (1) the mass flow rate in lbm/s and (2) the velocity in the nozzle jet. From the equation of state

$$\begin{aligned} v &= RT/p \\ v_D &= RT_D/p_D = 53.34\,(150 + 459.7)/(144 \times 20) \\ &= 11.29 \text{ ft}^3/\text{lbm} \\ v_J &= RT_J/p_J = 53.34\,(147 + 459.7)/(144 \times 14.7) \\ &= 15.29 \text{ ft}^3/\text{lbm} \\ (1)\ \dot{m} &= V_D A_D/v_D = 18\,[(\pi/4)(12/12)^2]/11.29 \\ \dot{m}_J &= 1.252 \text{ lbm/s } (0.5680 \text{ kg/s}) \\ (2)\ VJ &= mv_J/A_j = (1.252)(15.29)/[(\pi/4)(4/12)^2] \\ v_J &= 219.2 \text{ ft/s } (66.82 \text{ m/s}) \end{aligned}$$

FLUID DYNAMICS

Equation of Motion For steady one-dimensional flow, consideration of forces acting on a fluid element of length dL, flow area dA, boundary perimeter in fluid contact dP, and change in elevation dz with a unit shear stress τ moving at a velocity of V results in

$$v\, dp + \frac{V\, dV}{g_c} + \frac{g}{g_c}\, dz + v\tau\left(\frac{dP}{dA}\right) dL = 0$$

Substituting $v = g/g_c\gamma$ and simplifying,

$$\frac{dp}{\gamma} + \frac{V\, dV}{g} + dz + dh_f = 0$$

where $dh_f = (\tau/\gamma)(dP/dA)\, dL = \tau\, dL/\gamma R_h$.

The expression $1/(dP/dA)$ is the **hydraulic radius** R_h and equals the flow area divided by the perimeter of the solid boundary in contact with the fluid. This perimeter is usually called the ''wetted'' perimeter. The hydraulic radius of a pipe flowing full is $(\pi D^2/4)/\pi D = D/4$. Values for other configurations are given in Table 3.3.6. Integration of the equation of motion for an incompressible fluid results in

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_{1f2}$$

Each term of the equation is in feet and is equivalent to the height the fluid would rise in a tube if its energy were converted into potential energy. For this reason, in hydraulic practice, each type of energy is referred to as a **head**. The **static pressure head** is p/γ. The **velocity head** is $V^2/2g$, and the **potential head** is z. The energy loss between sections h_{1f2} is called the **lost head** or **friction head**. The **energy grade line** at any point $\Sigma(p/\gamma + V^2/2g + z)$, and the **hydraulic grade line** is $\Sigma(p/\gamma + z)$ as shown in Fig. 3.3.14.

EXAMPLE. A 12-in pipe (11.938 in inside diameter) reduces to a 6-in pipe (6.065 in inside diameter). Benzene at 68°F (20°C) flows steadily through this system. At section 1, the 12-in pipe centerline is 10 ft above the datum, and at section 2, the 6-in pipe centerline is 15 ft above the datum. The pressure at section 1 is 20 lbf/in^2 and the velocity is 4 ft/s. If the head loss due to friction is 0.05 $V_2^2/2g$, compute the pressure at section 2. Assume $g = g_c$, $\gamma = \rho g = 1.705 \times 32.17 = 54.85$ lbf/ft^3. From the continuity equation,

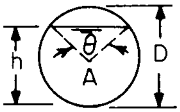
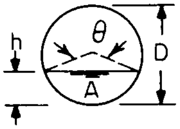
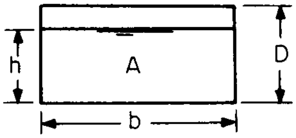
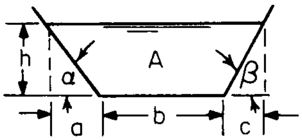
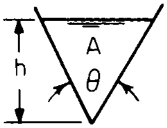
$$\begin{aligned} \dot{M} &= \rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad (p_1 = p_2) \\ V_2 &= V_1(A_1/A_2) = V_1(\pi D_1^2)/4)/(\pi D_2^2/4) = V_1(D_1/D_2)^2 \\ V_2 &= 4(11.938/6.065)^2 = 15.50 \text{ ft/s} \end{aligned}$$

From the equation of motion,

$$\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 - \left(\frac{V_2^2}{2g} + z_2 + h_{1f2}\right)$$

$$\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + \frac{V_1^2 - V_2^2 - 0.05V_2^2}{2g} + z_1 - z_2$$

Table 3.3.6 Values of Flow Area *A* and Hydraulic Radius R_h for Various Cross Sections

Cross section	Condition			Equations
	Flowing full	$h/D = 1$		$A = \pi D^2/4 \quad R_h = D/4$
	Upper half partly full	$0.5 < h/D < 1$		$\cos(\theta/2) = (2h/D - 1)$ $A = [\pi(360 - \theta) + 180 \sin\theta](D^2/1{,}440)$ $R_h = [1 + (180 \sin\theta)/(\pi\theta)]\,(D/4)$
		$h/D = 0.8128$		$A = 0.6839\, D^2\ R_h \max = 0.3043D$
	Lower half partly full	$h/D = 0.5$		$A = \pi D^2/8 \quad R_h \max = h/2$
		$0 < h/D < 0.5$		$\cos(\theta/2) = (1 - 2h/D)$ $A = (\pi\theta - 180 \sin\theta)\,(D^2/1{,}440)$ $R_h = [1 - (180 \sin\theta)/(\pi\theta)](D/4)$
	Flowing full	$h/D = 1$		$A = bD \quad R_h = bD/2(b + D)$
		Square $b = D$		$A = D^2 \quad R_h = D/4$
	Partly full	$h/D < 1$ $h/b = 0.5$ $b \rightarrow \infty,\ h \rightarrow 0$		$A = bh \quad R_h = bh/(2h + b)$ $A = b^2/2 \quad R_h \max = h/2$ $R_h \rightarrow h$ (wide shallow stream)
	$\alpha \neq \beta$			$R_h \max = h/2$ $A = [b + 1/2h(\cot\alpha + \cot\beta)]h$ $R_h = A/[b + h(\csc\alpha + \csc\beta)]$
	$\alpha = \beta$	$\frac{h}{a} = \frac{1}{2}$	$\alpha = 26°34'$	$A = (b + 2h)h$ $R_h = (b + 2h)h/(b + 4.472h)$
		$\frac{h}{a} = \frac{\sqrt{3}}{3}$	$\alpha = 30°$	$A = (b + 1.732h)h$ $R_h = (b + 1.732h)h/(b + 4h)$
		$\frac{h}{a} = \frac{2}{3}$	$\alpha = 33°41'$	$A = (b + 1.5h)h$ $R_h = (b + 1.5h)h/(b + 3.606h)$
		$\frac{h}{a} = 1$	$\alpha = 45°$	$A = (b + h)h$ $R_h = (b + h)h/(b + 2.828h)$
		$\frac{h}{a} = \frac{3}{2}$	$\alpha = 56°19'$	$A = (b + 0.6667h)h$ $R_h = (b + 0.6667h)h/(b + 2.404h)$
		$\frac{h}{a} = \sqrt{3}$	$\alpha = 60°$	$A = (b + 0.5774h)h$ $R_h = (b + 0.5774h)h/(b + 2.309h)$
	θ = any angle			$A = \tan(\theta/2)h^2 \quad R_h = \sin(\theta/2)h/2$
	$\theta = 30$			$A = 0.2679h^2 \quad R_h = 0.1294h$
	$\theta = 45$			$A = 0.4142h^2 \quad R_h = 0.1913h$
	$\theta = 60$			$A = 0.5774h^2 \quad R_h = 0.2500h$
	$\theta = 90$			$A = h^2 \quad R_h = 0.3536h$

$$\frac{p_2}{\gamma} = \frac{144 \times 20}{54.85} + \frac{4^2 - 1.05(15.50)^2}{2 \times 32.17} + 10 - 15$$

$$\frac{p_2}{\gamma} = 43.83 \text{ ft}$$

$$p_2 = \frac{54.88 \times 43.83}{144} = 16.70 \text{ lbf/in}^2\ (1.151 \times 10^5 \text{ N/m}^2)$$

Energy Equation Application of the principles of conservation of energy to a control volume for one-dimensional flow results in the following for steady flow:

$$J\,dq = dW + \frac{V\,dV}{g_c} + \frac{g}{g_c}\,dz + J\,du + d(pv)$$

where J is the mechanical equivalent of heat, 778.169 ft · lbf/Btu; q is the heat added, Btu/lbm (2,326 J/kg); W is the steady-flow shaft work done *by* the fluid; and u is the internal energy. Btu/lbm (2,326 J/kg). If the energy equation is integrated for an incompressible fluid,

$$J_1q_2 = {}_1W_2 + \frac{V_2^2 - V_1^2}{2g_c} + \frac{g}{g_c}(z_2 - z_1) + J(u_2 - u_1) + v(p_2 - p_1)$$

The equation of motion does not consider thermal energy or steady-flow work; the energy equation has no terms for friction. Subtracting the differential equation of motion from the energy equation and solving for friction results in

$$dh_f = (dW + J\ du + p\,dv - J\,dq)(g_c/g)$$

Integrating for an incompressible fluid ($dv = 0$),

$$h_{1f2} = [{}_1W_2 + J(u_2 - u_1) - J_1q_2](g_c/g)$$

In the absence of steady-flow work in the system, the effect of friction is to increase the internal energy and/or to transfer heat from the system.

For steady frictionless, incompressible flow, both the equation of motion and the energy equation reduce to

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

which is known as the **Bernoulli equation**.

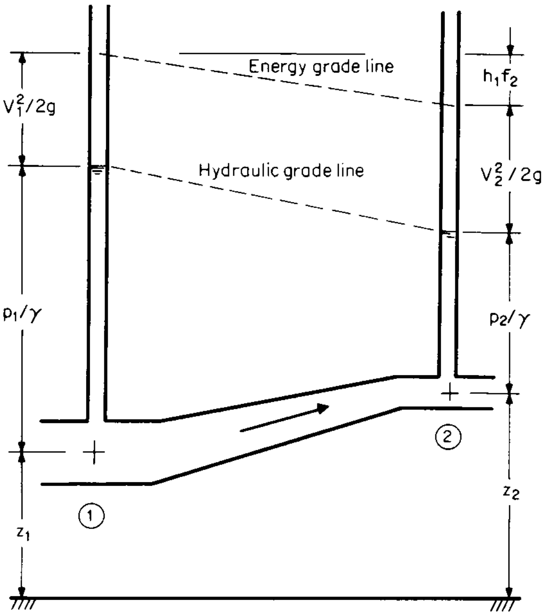


Fig. 3.3.14 Energy relations.

Area-Velocity Relations The continuity equation may be written as $\log_e \dot{M} = \log_e V + \log_e A + \log_e \rho$, which when differentiated becomes

$$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho}$$

For incompressible fluids, $d\rho = 0$, so

$$\frac{dA}{A} = -\frac{dV}{V}$$

Examination of this equation indicates

1. If the area increases, the velocity decreases.
2. If the area is constant, the velocity is constant.
3. There are no critical values.

For the frictionless flow of compressible fluids, it can be demonstrated that

$$\frac{dA}{A} = -\frac{dV}{V}\left[1 - \left(\frac{V}{c}\right)^2\right]$$

Analysis of the above equation indicates:

1. *Subsonic velocity* $V < c$. If the area increases, the velocity decreases. Same as for incompressible flow.

2. *Sonic velocity* $V = c$. Sonic velocity can exist only where the change in area is zero, i.e., at the *end* of a convergent passage or at the *exit* of a constant-area duct.

3. *Supersonic velocity* $V > c$. If area increases, the velocity increases, the reverse of incompressible flow. Also, supersonic velocity can exist only in the **expanding** portion of a passage **after** a constriction where **sonic** velocity existed.

Frictionless adiabatic compressible flow of an ideal gas in a horizontal passage must satisfy the following requirements:

1. *Conservation of mass*. As expressed by the continuity equation $M = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$.

2. *Conservation of energy*. As expressed by the energy equation

$$\frac{V^2}{2g_c} + Ju + pv = \frac{V_1^2}{2g_c} + Ju_1 + p_1v_1 = \frac{V_2^2}{2g_c} + Ju_2 + p_2v_2$$

3. *Process relationship*. For an ideal gas undergoing a frictionless adiabatic (isentropic) process,

$$pv^k = p_1v_1^k = p_2v_2^k$$

4. *Ideal-gas law*. The equation of state for an ideal gas

$$pv = RT$$

In an expanding supersonic flow, a **compression shock wave** will be formed if the requirements for the conservation of mass and energy are not satisfied. This type of wave is associated with large and sudden rises in pressure, density, temperature, and entropy. The shock wave is so thin that for computation purposes it may be considered as a single line. For compressible flow of gases and vapors in passages, refer to Sec. 4.1; for steam-turbine passages, Sec. 9.4; for compressible flow around immersed objects, see Sec. 11.4.

The **impulse-momentum equation** is an application of the principle of conservation of momentum and is derived from Newton's second law. It is used to calculate the forces exerted on a solid boundary by a moving stream. Because velocity and force have both magnitude and direction, they are vectors. The impulse-momentum equation may be written for all three directions:

$$\Sigma F_x = \dot{M}(V_{x2} - V_{x1})$$
$$\Sigma F_y = \dot{M}(V_{y2} - V_{y1})$$
$$\Sigma F_z = \dot{M}(V_{z2} - V_{z1})$$

Figure 3.3.15 shows a free-body diagram of a control volume. The pressure forces shown are those imposed by the boundaries on the fluid and on the atmosphere. The reactive force R is that imposed by the downstream boundary on the fluid for equilibrium. Application of the impulse-momentum equation yields

$$\Sigma F = (F_{p1} + F_{a2}) - (F_{a1} + F_{p2} + R) = \dot{M}(V_2 - V_1)$$

Solving for R,

$$R = (p_1 - p_a)A_1 - (p_2 - p_a)A_2 = \dot{M}(V_2 = V_1)$$

The impulse-momentum equation is often used in conjunction with the continuity and energy equations to solve engineering problems. Because of the wide variety of possible applications, some examples are given to illustrate the methods of attack.

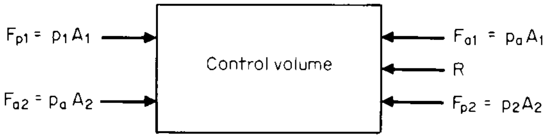


Fig. 3.3.15 Notation for impulse momentum.

EXAMPLE. **Compressible Fluid in a Duct.** Nitrogen flows steadily through a 6-in (5.761 in inside diameter) straight, horizontal pipe at a mass rate of 25 lbm/s. At section 1, the pressure is 120 lbf/in^2 and the temperature is 100°F. At section 2, the pressure is 80 lbf/in^2 and the temperature is 110°F. Find the friction force opposing the motion. From the equation of state,

$$v = RT/p$$
$$v_1 = 55.16\,(459.7 + 100)/(144 \times 120) = 1.787\ \text{ft}^3/\text{lbm}$$
$$v_2 = 55.16\,(459.7 + 110)/(144 \times 80) = 2.728\ \text{ft}^3/\text{lbm}$$

$$\text{Flow area of pipe} \times \pi D^2/4 = \pi(5.761/12)^2/4 = 0.1810\ \text{ft}^2$$

From the continuity equation,

$$v = \dot{m}V/A$$
$$V_1 = (25 \times 1.787)/0.1810 = 246.8\ \text{ft/s}$$
$$V_2 = (25 \times 2.728)/0.1810 = 376.8\ \text{ft/s}$$

Applying the free-body equation for impulse momentum ($A = A_1 = A_2$),

$$\begin{aligned} R &= (p_1 - p_a)\,A_1 - (p_2 - p_a)\,A_2 - \dot{M}(V_2 - V_1) \\ &= (p_1 - p_2)\,A - M(V_2 - V_1) = 144\ (120 - 80)\ 0.1810 \\ &\quad - (25/32.17)(376.8 - 246.8) = 941.5\ \text{lbf}\ (4.188 \times 10^3\ \text{N}) \end{aligned}$$

EXAMPLE. **Water Flow through a Nozzle.** Water at 68°F (20°C) flows through a horizontal 12- by 6-in-diameter nozzle discharging into the atmosphere. The pressure at the nozzle inlet is 65 lbf/in² and barometric pressure is 14.7 lbf/in². Determine the force exerted by the water on the nozzle.

$$\begin{aligned} A &= \pi D^2/4 \\ A_1 &= \pi(12/12)^2/4 = 0.7854\ \text{ft}^2 \\ A_2 &= \pi(6/12)^2/4 = 0.1963\ \text{ft}^2 \\ \gamma &= \rho g = 1.937 \times 32.17 = 62.31\ \text{lbf/ft}^3 \end{aligned}$$

From the continuity equation $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ for $\rho_1 = \rho_2$, $V_2 = V_1 A_1/A_2 = (0.7854/0.1963)V_1 = 4V_1$. From Bernoulli's equation ($z_1 = z_2$).

$$p_1/\gamma + V_1^2/2g = p_2/\gamma + V_2^2/2g = p_2/\gamma + (4V_1)^2/2g$$

or

$$\begin{aligned} V_1 &= \sqrt{2g(p_1 - p_2)/15\gamma} \\ &= \sqrt{2 \times 32.17 \times 144\ (65 - 14.7)/15 \times 62.31} = 22.33\ \text{ft/s} \\ V_2 &= 4 \times 22.33 = 89.32\ \text{ft/s} \end{aligned}$$

Again from the equation of continuity

$$\dot{M} = \rho_1 A_1 V_1 = 1.937 \times 0.7854 \times 22.33 = 33.97\ \text{slugs/s}$$

Applying the free-body equation for impulse momentum,

$$\begin{aligned} R &= (p_1 - p_a)\,A_1 - (p_2 - p_a)\,A_2 - \dot{M}(V_2 - V_1) \\ &= 144\ (65 - 14.7)\ 0.7854 - 144\ (14.7 - 14.7)\ 0.1963 \\ &= (33.97)\ (89.32 - 22.33) \\ &= 3{,}413\ \text{lbf}\ (1.518 \times 10^4\ \text{N}) \end{aligned}$$

EXAMPLE. **Incompressible Flow through a Reducing Bend.** Carbon tetrachloride flows steadily without friction at 68°F (20°C) through a 90° horizontal reducing bend. The mass flow rate is 4 slugs/s, the inlet diameter is 6 in, and the outlet is 3 in. The inlet pressure is 50 lbf/in² and the barometric pressure is 14.7 lbf/in². Compute the magnitude and direction of the force required to ''anchor'' this bend.

$$\begin{aligned} A &= \pi D^2/4 \\ A_1 &= (\pi/4)(6/12)^2 = 0.1963\ \text{ft}^2 \\ A_2 &= (\pi/4)(3/12)^2 = 0.04909\ \text{ft}^2 \end{aligned}$$

From continuity,

$$\begin{aligned} V &= M/\rho A \\ V_1 &= 4/(3.093)(0.1963) = 6.588\ \text{ft/s} \\ V_2 &= 4/(3.093)(0.04909) = 26.35\ \text{ft/s} \end{aligned}$$

From the Bernoulli equation ($z_1 = z_2$),

$$\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = \frac{144 \times 50}{3.093 \times 32.17} + \frac{(6.588)^2 - (26.35)^2}{2 \times 32.17} = 62.24\ \text{ft}$$

$$p_2 = \frac{(3.093 \times 32.17)(62.24)}{144} = 43.01\ \text{lbf/in}^2$$

From Fig. 3.3.16,

$$\begin{aligned} \Sigma F_x &= (p_1 - p_a)A_1 - (p_2 - p_a)A_2 \cos\alpha - R_x \\ &= M(V_2 \cos\alpha - V_1) \end{aligned}$$

or

$$\begin{aligned} R_x &= (p_1 - p_a)A_1 - (p_2 - p_a)A_2 \cos\alpha - \dot{M}(V_2 \cos\alpha - V_1) \\ \Sigma F_y &= 0 - (p_2 - p_a)A_2 \sin\alpha + R_y = M(V_2 \sin\alpha - 0) \end{aligned}$$

or

$$\begin{aligned} R_y &= (p_2 - p_a)A_2 \sin\alpha + MV_2 \sin\alpha \\ R_x &= 144\ (50 - 14.7)\ 0.1963 - 144\ (43.01 - 14.7)(\cos\ 90°) \\ &\quad - 4\ (26.35 \cos 90° - 6.588) \\ &= 1{,}024\ \text{lbf} \\ R_y &= 144\ (43.01 - 14.7)(0.04909) \sin 90° + 4(26.35)(\sin 90°) \\ &= 305.5\ \text{lbf} \\ R &= \sqrt{R_x^2 + R_y^2} = \sqrt{(1{,}024)^2 + (305.5)^2} \\ &= 1{,}068\ \text{lbf}_f(4.753 \times 10^3\ \text{N}) \\ \theta &= \tan^{-1}(F_y/F_x) = \tan^{-1}(305.5/1{,}024) \\ &= 16°37' \end{aligned}$$

Forces on Blades and Deflectors The forces imposed on a fluid jet whose velocity is V_J by a blade moving at a speed of V_b away from the jet are shown in Fig. 3.3.17. The following equations were developed from the application of the impulse-momentum equation for an open jet ($p_2 = p_1$) and for **frictionless** flow:

$$\begin{aligned} F_x &= \rho A_J(V_J - V_b)^2(1 - \cos\alpha) \\ F_y &= \rho A_J(V_J - V_b)^2 \sin\alpha \\ F &= 2\rho A_J(V_J - V_b)^2 \sin(\alpha/2) \end{aligned}$$

EXAMPLE. In the nozzle-blade system of Fig. 3.3.17, water at 68°F (20°C) enters a 3- by 1½-in-diameter horizontal nozzle with a pressure 23 lbf/in² and discharges at 14.7 lbf/in² (atmospheric pressure). The blade moves away from the nozzle at a velocity of 10 ft/s and deflects the stream through an angle of 80°. For frictionless flow, calculate the total force exerted by the jet on the blade. Assume $g = g_c$; then $\gamma = \rho g$. From the continuity equation ($\rho_I = \rho_J$), $\rho_I A_I V_I = \rho_J A_J V_J$, $V_I = (A_J/A_I)V_J$,

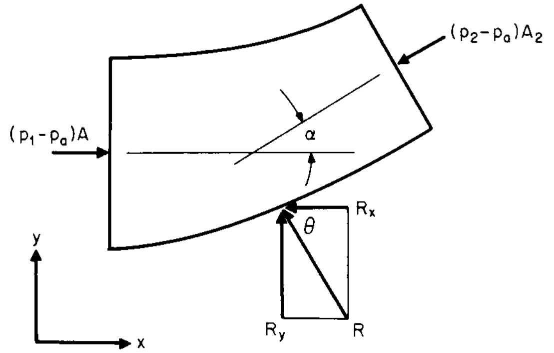


Fig. 3.3.16 Forces on a bend.

$$V_J = \frac{\pi D_J^2/4}{\pi D_I^2/4} V_J = \left(\frac{D_J}{D_I}\right)^2 V_J$$

$$V_I = (1.5/3)^2 V_J = V_J/4$$

From the Bernoulli equation ($z_2 = z_1$),

$$\frac{V_J^2}{2g} = \frac{V_I^2}{2g} + \frac{p_I - p_J}{\rho g}$$

$$\frac{V_J^2 - (V_J/4)^2}{2g} = \frac{(p_I - p_J)}{\rho g}$$

$$\begin{aligned} V_J &= \sqrt{\frac{2(16/15)(p_I - p_J)}{\rho}} \\ &= \sqrt{\frac{2 \times (16/15)\ 144\ (23 - 14.7)}{1.937}} = 36.28\ \text{ft/s} \end{aligned}$$

The total force $F = 2\rho A_J(V_J - V_b)^2 \sin(\alpha/2)$

$$\begin{aligned} F &= 2 \times 1.937\ (\pi/4)(1.5/12)^2(36.28 - 10)^2 \sin(80/2) \\ &= 21.11\ \text{lbf}\ (93.90\ \text{N}) \end{aligned}$$

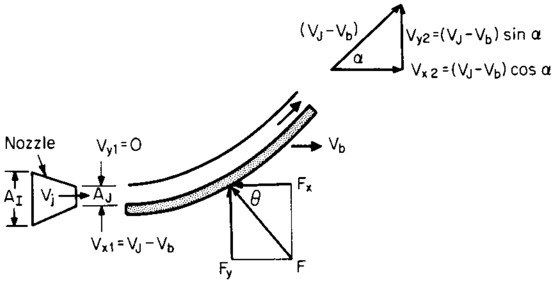


Fig. 3.3.17 Notation for blade study.

Impulse Turbine In a turbine, the total of the separate forces acting simultaneously on each blade equals that caused by the combined mass flow rate $\dot{M}$ discharged by the nozzle or

$$\Sigma P = \Sigma F_x V_b = \dot{M}(V_J - V_b)(1 - \cos\alpha)V_b$$

The maximum value of power P is found by differentiating P with respect to V_b and setting the result equal to zero. Solving for V_b yields $V_b = V_J/2$, so that maximum power occurs when the velocity of the jet is equal to twice the velocity of the blade. Examination of the power equation also indicates that the angle α for a maximum power results when $\cos\theta = -1$ or $\alpha = 180°$. For theoretical maximum power of a blade, $2V_b = V_J$ and $\alpha = 180°$. It should be noted that in any practical impulse-turbine application, α cannot be 180° because the discharge interferes with the next set of blades. Substituting $V_b = V_J/2$, $\alpha = 180°$ in the power equation,

$$\Sigma P_{\max} = \dot{M}(V_J - V_J/2)[1 - (-1)]V_J/2 = MV_J^2/2 = \dot{m}V_J^2/2g_c$$

or the maximum power per unit mass is equal to the total power of the jet. Application of the Bernoulli equation between the surface of a reservoir and the discharge of the turbine shows that $\Sigma P_{\max} = \dot{M}\sqrt{2g(z_2 - z_1)}$. For design details, see Sec. 9.9.

Flow in a Curved Path When a fluid flows through a bend, it is also rotated around an axis and the energy required to produce rotation must be supplied from the energy already in the fluid mass. This fluid rotation is called a **free vortex** because it is free of outside energy. Consider the fluid mass $\rho(r_o - r_i)\,dA$ of Fig. 3.3.18 being rotated as it flows through a bend of outer radius r_o, inner radius r_i, with a velocity of V. Application of Newton's second law to this mass results in

$$dF = p_o\,dA - p_i\,dA = [\rho(r_o - r_i)\,dA][V^2/(r_o + r_i)/2]$$

which reduces to

$$p_o - p_i = 2(r_o - r_i)\rho V^2/(r_o + r_i)$$

Because of the difference in fluid pressure between the inner and outer walls of the bend, secondary flows are set up, and this is the primary cause of friction loss of bends. These secondary flows set up turbulence that require 50 or more straight pipe diameters downstream to dissipate.

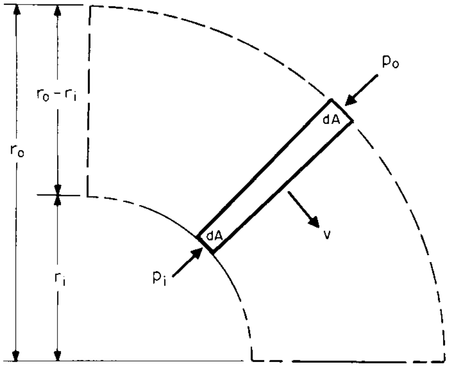


Fig. 3.3.18 Notation for flow in a curved path.

Thus this loss does not take place in the bend, but in the downstream system. These losses may be reduced by the use of splitter plates which help minimize the secondary flows by reducing $r_o - r_i$ and hence $p_o - p_i$.

EXAMPLE. 104°F (40°C) benzene flows at a rate of 8 ft^3/s in a square horizontal duct. This duct makes a 90° turn with an inner radius of 1 ft and an outer radius of 2 ft. Calculate the difference between the walls of this bend. The area of this duct is $(r_o - r_i)^2 = (2 - 1)^2 = 1\ \text{ft}^2$. From the continuity equation $V = Q/A = 8/1 = 8$ ft/s. The pressure difference

$$\begin{aligned} p_o - p_i &= 2(r_o - r_i)\rho V^2/(r_o + r_i) \\ &= 2(2 - 1)\ 1.663\ (8)^2/(2 + 1) = 70.95\ \text{lbf/ft}^2 \\ &= 70.95/144 = 0.4927\ \text{lbf/in}^2\ (3.397 \times 10^3\ \text{N/m}^2) \end{aligned}$$

DIMENSIONLESS PARAMETERS

Modern engineering practice is based on a combination of theoretical analysis and experimental data. Often the engineer is faced with the necessity of obtaining practical results in situations where for various reasons, physical phenomena cannot be described mathematically and experimental data must be considered. The generation and use of **dimensionless parameters** provides a powerful and useful tool in (1) reducing the number of variables required for an experimental program, (2) establishing the principles of model design and testing, (3) developing equations, and (4) converting data from one system of units to another. Dimensionless parameters may be generated from (1) **physical equations,** (2) the principles of **similarity,** and (3) **dimensional analysis.** All **physical equations** must be dimensionally correct so that a dimensionless parameter may be generated by simply dividing one side of the equation by the other. A minimum of two dimensionless parameters will be formed, one being the inverse of the other.

EXAMPLE. It is desired to generate a series of dimensionless parameters to describe the ratios of static pressure head, velocity head, and potential head to total head for frictionless incompressible flow. From the Bernoulli equation,

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \Sigma h = \text{total head}$$

$$N_1 = \frac{p/\gamma + V^2/2g + z}{\Sigma h} = \frac{p/\gamma}{\Sigma h} + \frac{V^2/2g}{\Sigma h} + \frac{z}{\Sigma h} = N_p + N_V + N_z$$

or

$$N_2 = \frac{\Sigma h}{p_2/\gamma + V^2/2g + z} = N_1^{-1}$$

N_1 and N_2 are total energy ratios and N_p, N_V, and N_z are the ratios of the static pressure head, velocity head, and potential head, respectively, to the total head.

Models versus Prototypes There are times when for economic or other reasons it is desirable to determine the performance of a structure or machine by testing another structure or machine. This type of testing is called model testing. The equipment being tested is called a **model,** and the equipment whose performance is to be predicted is called a **prototype.** A **model** may be smaller than, the same size as, or larger than the **prototype.** Model experiments on aircraft, rockets, missiles, pipes, ships, canals, turbines, pumps, and other structures and machines have resulted in savings that more than justified the expenditure of funds for the design, construction, and testing of the model. In some situations, the model and the prototype may be the same piece of equipment, for example, the laboratory calibration of a flowmeter with water to predict its performance with other fluids. Many manufacturers of fluid machinery have test facilities that are limited to one or two fluids and are forced to test with what they have available in order to predict performance with nonavailable fluids. For towing-tank testing of ship models and for wind-tunnel testing of aircraft and aircraft-component models, see Secs. 11.4 and 11.5.

Similarity Requirements For complete similarity between a model and its prototype, it is necessary to have **geometric, kinematic,** and **dynamic** similarity. **Geometric similarity** exists between model and prototype when the **ratios** of all corresponding dimensions of the model and prototype are equal. These ratios may be written as follows:

Length: $L_{model}/L_{prototype} = L_{ratio} = L_m/L_p = L_r$

Area: $L^2_{model}/L^2_{prototype} = L^2_{ratio} = L^2_m/L^2_p = L^2_r$

Volume: $L^3_{model}/L^3_{prototype} = L^3_{ratio} = L^3_m/L^3_p = L^3_r$

Kinematic similarity exists between model and prototype when their streamlines are geometrically similar. The kinematic ratios resulting from this condition are

Acceleration: $a_r = a_m/a_p = L_mT_m^{-2}/L_pT_p^{-2} = L_r/T_r^{-2}$

Velocity: $V_r = V_m/V_p = L_mT_m^{-1}/L_pT_p^{-1} = L_r/T_r^{-1}$

Volume flow rate: $Q_r = Q_m/Q_p = L^3_mT_m^{-1}/L^3_pT_p^{-1} = L^3_r/T_r^{-1}$

Dynamic similarity exists between model and prototype having geometric and kinematic similarity when the ratios of all forces are the same. Consider the model/prototype relations for the flow around the object shown in Fig. 3.3.19. For **geometric similarity** $D_m/D_p = L_m/L_p = L_r$ and for **kinematic similarity** $U_{Am}/U_{Ap} = U_{Bm}/U_{Bp} = V_r =$

$L_rT_r^{-1}$. Next consider the three forces acting on point C of Fig. 3.3.19 without specifying their nature. From the geometric similarity of their vector polygons and Newton's law, for **dynamic similarity** $F_{1m}/F_{1p} = F_{2m}/F_{2p} = F_{3m}/F_{3p} = M_m a_{Cm}/M_p a_{Cp} = F_r$. For dynamic similarity, these force ratios must be maintained on all corresponding fluid particles throughout the flow pattern. From the force polygon of Fig. 3.3.19, it is evident that $F_1 + \rightarrow F_2 + \rightarrow F_3 = Ma_C$. For total model/prototype force ratio, comparisons of force polygons yield

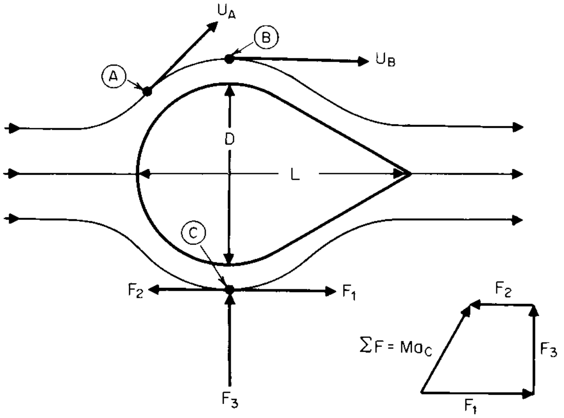


Fig. 3.3.19 Notation for dynamic similarity.

$$F_r = \frac{F_{1m} + \rightarrow F_{2m} + \rightarrow F_{3m}}{F_{1p} + \rightarrow F_{2p} + \rightarrow F_{3p}} = \frac{M_m a_{Cm}}{M_p a_{Cp}}$$

Fluid Forces The fluid forces that are considered here are those acting on a fluid element whose mass $= \rho L^3$, area $= L^2$, length $= L$, and velocity $= L/T$.

Inertia force $F_i = \text{(mass)(acceleration)} = (\rho L^3)(L/T^2) = \rho L(L^2/T^2) = \rho L^2V^2$

Viscous force $F_\mu = \text{(viscous shear stress)(shear area)} = \tau L^2 = \mu(dU/dy)L^2 = \mu(V/L)L^2 = \mu LV$

Gravity force $F_g = \text{(mass)(acceleration due to gravity)} = (\rho L^3)(g) = \rho L^3 g$

Pressure force $F_p = \text{(pressure)(area)} = pL^2$

Centrifugal force $F_\omega = \text{(mass)(acceleration)} = (\rho L^3)(L/T^2) = (\rho L^3)(L\omega^2) = \rho L^4\omega^2$

Elastic force $F_E = \text{(modulus of elasticity)(area)} = EL^2$

Surface-tension force $F_\sigma = \text{(surface tension)(length)} = \sigma L$

Vibratory force $F_f = \text{(mass)(acceleration)} = (\rho L^3)(L/T^2) = (\rho L^4)(T^{-2}) = \rho L^4 f^2$

If all fluid forces were acting on a fluid element,

$$F_r = \frac{F_{\mu m} + \rightarrow F_{gm} + \rightarrow F_{pm} + \rightarrow F_{\omega m} + \rightarrow F_{Em} + \rightarrow F_{\sigma m} + \rightarrow F_{fm}}{F_{\mu p} + \rightarrow F_{gp} + \rightarrow F_{pp} + \rightarrow F_{\omega p} + \rightarrow F_{Ep} + \rightarrow F_{\sigma p} + \rightarrow F_{fp}} = \frac{F_{im}}{F_{ip}}$$

Examination of the above equation and the force polygon of Fig. 3.3.19 lead to the conclusion that dynamic similarity can be characterized by an equality of force ratios one less than the total number involved. Any force ratio may be eliminated, depending upon the quantities which are desired. Fortunately, in most practical engineering problems, not all of the eight forces are involved because some may not be acting, may be of negligible magnitude, or may be in opposition to each other in such a way as to compensate. In each application of similarity, a good understanding of the fluid phenomena involved is necessary to eliminate the irrelevant, trivial, or compensating forces. When the flow phenomenon is too complex to be readily analyzed, or is not known, only experimental verification with the prototype of results from a model test will determine what forces should be considered in future model testing.

Standard Numbers With eight fluid forces that can act in flow situations, the number of dimensionless parameters that can be formed from

Table 3.3.7 Standard Numbers

			Conventional practice		
Force ratio	Equations	Result	Form	Symbol	Name
$\frac{\text{Inertia}}{\text{Viscous}}$	$\frac{F_i}{F_\mu} = \frac{\rho L^2V^2}{\mu LV}$	$\frac{\rho LV}{\mu}$	$\frac{\rho LV}{\mu}$	**R**	Reynolds
$\frac{\text{Inertia}}{\text{Gravity}}$	$\frac{F_i}{F_g} = \frac{\rho L^2V^2}{\rho L^3 g}$	$\frac{V^2}{Lg}$	$\frac{V}{\sqrt{Lg}}$	**F**	Froude
$\frac{\text{Inertia}}{\text{Pressure}}$	$\frac{F_i}{F_p} = \frac{\rho L^2V^2}{\rho L^2}$	$\frac{\rho V^2}{p}$	$\frac{\rho V^2}{p}$	**E**	Euler
			$\frac{2\Delta p}{\rho V^2}$	$\mathbf{C}_p$	Pressure coefficient
$\frac{\text{Inertia}}{\text{Centrifugal}}$	$\frac{F_i}{F_\omega} = \frac{\rho L^2V^2}{\rho L^4\omega^2}$	$\frac{V^2}{L^2\omega^2}$	$\frac{V}{DN}$	**V**	Velocity ratio
$\frac{\text{Inertia}}{\text{Elastic}}$	$\frac{F_i}{F_E} = \frac{\rho L^2V^2}{EL^2}$	$\frac{\rho V^2}{E}$	$\frac{\rho V^2}{E}$	C	Cauchy
			$\frac{V}{\sqrt{E/\rho}}$	**M**	Mach
$\frac{\text{Inertia}}{\text{Surface tension}}$	$\frac{F_i}{F_\sigma} = \frac{\rho L^2V^2}{\sigma^L}$	$\frac{\rho LV^2}{\sigma}$	$\frac{\rho LV^2}{\sigma}$	**W**	Weber
$\frac{\text{Inertia}}{\text{Vibration}}$	$\frac{F_i}{F_f} = \frac{\rho L^2V^2}{\rho L^4 f^2}$	$\frac{V^2}{L^2f^2}$	$\frac{Lf}{V}$	**S**	Strouhal

SOURCE: Computed from data given in Murdock, ''Fluid Mechanics and Its Applications,'' Houghton Mifflin, 1976.

their ratios is 56. However, conventional practice is to ratio the inertia force to the other fluid forces, usually by division because the inertia force is the vector sum of all the other forces involved in a given flow situation. Results obtained by dividing the inertia force by each of the other forces are shown in Table 3.3.7 compared with the standard numbers that are used in conventional practice.

DYNAMIC SIMILARITY

Vibration In the flow of fluids around objects and in the motion of bodies immersed in fluids, **vibration** may occur because of the formation of a wake caused by alternate shedding of eddies in a periodic fashion or by the vibration of the object or the body. The **Strouhal number S** is the ratio of the velocity of vibration Lf to the velocity of the fluid V. Since the vibration may be fluid-induced or structure-induced, two frequencies must be considered, the wake frequency $f\omega$ and the natural frequency of the structure f_n. Fluid-induced forces are usually of small magnitude, but as the wake frequency approaches the natural frequency of the structure, the vibratory forces increase very rapidly. When $f\omega = f_n$, the structure will go into resonance and fail. This imposes on the model designer the requirement of matching to scale the natural-frequency characteristics of the prototype. This subject is treated later under Wake Frequency. All further discussions of model/prototype relations are made under the assumption that either vibratory forces are absent or they are taken care of in the design of the model or in the test program.

Incompressible Flow Considered in this category are the flow of fluids around an object, motion of bodies immersed in incompressible fluids, and the flow of incompressible fluids in conduits. It includes, for example, a submarine traveling under water but not partly submerged, and liquids flowing in pipes and passages when the liquid completely fills them, but not when partly full as in open-channel flow. It also includes aircraft moving in atmospheres that may be considered incompressible. Incompressible flow in rotating machinery is considered separately.

In these situations the gravity force, although acting on all fluid particles, does not affect the flow pattern. Excluding rotating machinery, centrifugal forces are absent. By definition of an incompressible fluid, elastic forces are zero, and since there is no liquid-gas interface, surface-tension forces are absent.

The only forces now remaining for consideration are the inertia, viscous, and pressure. Using standard numbers, the parameters are **Reynolds number** and **pressure coefficient**. The Reynolds number may be converted into a kinematic ratio by noting that by definition $v = \mu/\rho$ and substituting in $\mathbf{R} = \rho LV/\mu = LV/v$. In this form, Reynolds number is the ratio of the fluid velocity V and the ''shear velocity'' v/L. For this reason, Reynolds number is used to characterize the velocity profile. Forces and pressure losses are then determined by the pressure coefficient.

EXAMPLE. A submarine is to move submerged through 32°F (0°C) seawater at a speed of 10 knots. (1) At what speed should a 1 : 20 model be towed in 68°F (20°C) fresh water? (2) If the thrust on the model is found to be 42,500 lbf, what horsepower will be required to propel the submarine?

1. Speed of model for Reynolds-number similarity

$$R_m = R_p = \left(\frac{\rho VL}{\mu}\right)_m = \left(\frac{\rho VL}{\mu}\right)_p$$

$$\begin{aligned}V_m &= V_p(\rho_p/\rho_m)(L_p/L_m)(\mu_m/\mu_p)\\ V_m &= (10)(1.995/1.937)(20/1)(20.92 \times 10^{-6}/39.40 \times 10^{-6})\\ &= 109.4 \text{ knots } (56.27 \text{ m/s})\end{aligned}$$

2. Prototype horsepower

$$\mathbf{C}_{pp} = \mathbf{C}_{pm} = \left(\frac{2\Delta p}{\rho V^2}\right)_p = \left(\frac{2\Delta p}{\rho V^2}\right)_m$$

$$F = \Delta p L^2,\ \Delta p = \frac{F}{L^2}, \text{ so that } \left(\frac{2\,F}{\rho V^2 L^2}\right)_p = \left(\frac{2\,F}{\rho V^2 L^2}\right)_m$$

$$\begin{aligned}F_p &= F_m(\rho_p/\rho_m)(V_p/V_m)^2(L_p/L_m)^2\\ &= 42{,}500\ (1.995/1.937)(10/109.4)^2(20/1)^2 = 146{,}300 \text{ lbf}\end{aligned}$$

$$\begin{aligned}P = FV &= \left(\frac{146{,}300}{550}\right)\left(\frac{10 \times 6{,}076}{3{,}600}\right)\\ &= 4{,}490 \text{ hp } (3.35 \times 10^6 \text{ W})\end{aligned}$$

Compressible Flow Considered in this category are the flow of compressible fluids under the conditions specified for incompressible flow in the preceding paragraphs. In addition to the forces involved in incompressible flow, the elastic force must be added. Conventional practice is to use the square root of the inertia/elastic force ratio or **Mach number.**

Mach number is the ratio of the fluid velocity to its speed of sound and may be written $\mathbf{M} = V/c = V\sqrt{E_s/\rho}$. For an ideal gas, $\mathbf{M} = V/\sqrt{kg_cRT}$. In compressible-flow problems, practice is to use the Mach number to characterize the velocity or kinematic similarity, the Reynolds number for dynamic similarity, and the pressure coefficient for force or pressure-loss determination.

EXAMPLE. An airplane is to fly at 500 mi/h in an atmosphere whose temperature is 32°F (0°C) and pressure is 12 lbf/in². A 1 : 20 model is tested in a wind tunnel where a supply of air at 392°F (200°C) and variable pressure is available. At (1) what speed and (2) what pressure should the model be tested for dynamic similarity?

1. *Speed for Mach-number similarity*

$$\mathbf{M}_m = \mathbf{M}_p = \left(\frac{V}{\sqrt{E/\rho}}\right)_m = \left(\frac{V}{\sqrt{E/\rho}}\right)_p = \left(\frac{V}{\sqrt{kg_cRT}}\right)_m = \left(\frac{V}{\sqrt{kg_cRT}}\right)_p$$

$$V_m = V_p(k_m/k_p)^{1/2}(R_m/R_p)^{1/2}(T_m/T_p)^{1/2}$$

For the same gas $k_m = k_p$, $R_m = R_p$, and

$$V_m = V_p\sqrt{T_m/T_p} = 500\sqrt{(851.7/491.7)} = 658.1 \text{ mi/h}$$

2. *Pressure for Reynolds-number similarity*

$$\mathbf{R}_m = \mathbf{R}_p = \left(\frac{\rho VL}{\mu}\right)_m = \left(\frac{\rho VL}{\mu}\right)_p$$

$$\rho_m = \rho_p(V_p/V_m)(L_p/L_m)(\mu_m/\mu_p)$$

Since $\rho = p/g_cRT$

$$\left(\frac{p}{g_cRT}\right)_m = \left(\frac{p}{g_cRT}\right)_p (V_p/V_m)(L_p/L_m)(\mu_m/\mu_p)$$

$$\begin{aligned}p_m &= p_p(T_m/T_p)(V_p/V_m)(L_p/L_m)(\mu_m/\mu_p)\\ p_m &= 12(851.7/491.7)(500/658.1)(20/1)(53.15 \times 10^{-6}/35.67 \times 10^{-6})\\ p_m &= 470.6 \text{ lbf/in}^2\ (3.245 \times 10^6 \text{ N/m}^2)\end{aligned}$$

For information about wind-tunnel testing and its limitations, refer to Sec. 11.4.

Centrifugal Machinery This category includes the flow of fluids in such centrifugal machinery as compressors, fans, and pumps. In addition to the inertia, pressure, viscous, and elastic forces, centrifugal forces must now be considered. Since centrifugal force is really a special case of the inertia force, their ratio as shown in Table 3.3.7 is **velocity ratio** and is the ratio of the fluid velocity to the machine tangential velocity. In model/prototype relations for centrifugal machinery, DN (D = diameter, ft, N = rotational speed) is substituted for the velocity V, and D for L, which results in the following:

$$\mathbf{M} = DN/\sqrt{kg_cRT} \qquad \mathbf{R} = \rho D^2N/\mu \qquad \mathbf{C}_p = 2\Delta p/\rho D^2N^2$$

EXAMPLE. A centrifugal compressor operating at 100 r/min is to compress methane delivered to it at 50 lbf/in² and 68°F (20°C). It is proposed to test this compressor with air from a source at 140°F (60°C) and 100 lbf/in². Determine compressor speed and inlet-air pressure required for dynamic similarity.

Find speed for Mach-number similarity:

$$\begin{aligned}\mathbf{M}_m = \mathbf{M}_p &= (DN/\sqrt{kg_cRT})_m = (DN/\sqrt{kg_cRT})_p\\ N_m &= N_p(D_p/D_m)\sqrt{(k_m/k_p)(R_m/R_p)(T_m/T_p)}\\ &= 100\ (1)\sqrt{(1.40/1.32)(53.34/96.33)(599.7/527.7)}\\ &= 81.70 \text{ r/min}\end{aligned}$$

Find pressure for Reynolds-number similarity:

$$\mathbf{R}_m = \mathbf{R}_p = \left(\frac{\rho D^2N}{\mu}\right)_m = \left(\frac{\rho D^2N}{\mu}\right)_p$$

For an ideal gas $\rho = p/g_cRT$, so that

$$(pD^2N/g_cRT\mu)_m = (pD^2N/g_cRT\mu)_p$$

$$\begin{aligned} p_m &= p_p(D_p/D_m)^2(N_p/N_m)(R_m/R_p)(T_m/T_p)(\mu_m/\mu_p) \\ &= 50(1)^2(100/81.70)(53.34/96.33)(599.7/527.7) \times (41.79 \\ &\quad \times 10^{-8}/22.70 \times 10^{-8}) = 70.90 \text{ lbf/in}^2 \ (4.888 \times 10^5 \text{ N/m}^2) \end{aligned}$$

See Sec. 14 for specific information on pump and compressor similarity.

Liquid Surfaces Considered in this category are ships, seaplanes during takeoff, submarines partly submerged, piers, dams, rivers, open-channel flow, spillways, harbors, etc. Resistance at liquid surfaces is due to surface tension and wave action. Since wave action is due to gravity, the gravity force and surface-tension force are now added to the forces that were considered in the last paragraph. These are expressed as the square root of the inertia/gravity force ratio or **Froude number** $\mathbf{F} = V/\sqrt{Lg}$ and as the inertia/surface tension force ratio or **Weber number** $\mathbf{W} = \rho LV^2/\sigma$. On the other hand, elastic and pressure forces are now absent. Surface tension is a minor property in fluid mechanics and it normally exerts a negligible effect on wave formation *except* when the waves are small, say less than 1 in. Thus the effects of surface tension on the model might be considerable, but negligible on the prototype. This type of ''scale effect'' must be avoided. For accurate results, the inertia/surface tension force ratio or Weber number should be considered. It is never possible to have complete dynamic similarity of liquid surfaces unless the model and prototype are the same size, as shown in the following example.

EXAMPLE. An ocean vessel 500 ft long is to travel at a speed of 15 knots. A 1 : 25 model of this ship is to be tested in a towing tank using seawater at design temperature. Determine the model speed required for (1) wave-resistance similarity, (2) viscous or skin-friction similarity, (3) surface-tension similarity, and (4) the model size required for complete dynamic similarity.

1. *Speed for* ***Froude-number*** *similarity*

$$\mathbf{F}_m = \mathbf{F}_p = (V/\sqrt{Lg})_m = (V/\sqrt{Lg})_p$$

or

$$V_m = V_p\sqrt{L_m/L_p} = 15\sqrt{1/25} = 3 \text{ knots}$$

2. *Speed for* ***Reynolds-number*** *similarity*

$$\begin{aligned} \mathbf{R}_m &= \mathbf{R}_p = (\rho LV/\mu)_m = (\rho LV/\mu)_p \\ V_m &= V_p(\rho_p/\rho_m)(L_p/L_m)(\mu_m/\mu_p) \\ V_m &= 15(1)(25/1)(1) = 375 \text{ knots} \end{aligned}$$

3. *Speed for* ***Weber-number*** *similarity*

$$\begin{aligned} \mathbf{W}_m &= \mathbf{W}_p = (\rho LV^2/\sigma)_m = (\rho LV^2/\sigma)_p \\ V_m &= V_p\sqrt{(\rho_p/\rho_m)(L_p/L_m)(\sigma_m/\sigma_p)} \\ V_m &= 15\sqrt{(1)(25)(1)} = 75 \text{ knots} \end{aligned}$$

4. *Model size for complete similarity.* First try Reynolds and Froude similarity; let

$$V_m = V_p(\rho_p/\rho_m)(L_p/L_m)(\mu_m/\mu_p) = V_p\sqrt{L_m/L_p}$$

which reduces to

$$L_m/L_p = (\rho_p/\rho_m)^{2/3}(\mu_m/\mu_p)^{2/3}$$

Next try Weber and Froude similarity; let

$$V_m = V_p\sqrt{(\rho_p/\rho_m)(L_p/L_m)(\sigma_m/\sigma_p)} = V_p\sqrt{L_m/L_p}$$

which reduces to

$$L_m/L_p = (\rho_p/\rho_m)^{1/2}(\sigma_m/\sigma_p)^{1/2}$$

For the same fluid at the same temperature, either of the above solves for $L_m = L_p$, or the model must be the same size as the prototype. For use of different fluids and/or the same fluid at different temperatures.

$$L_m/L_p = (\rho_p/\rho_m)^{2/3}(\mu_m/\mu_p)^{2/3} = (\rho_p/\rho_m)^{1/2}(\sigma_m/\sigma_p)^{1/2}$$

which reduces to

$$(\mu^4/\rho\sigma^3)_m = (\mu^4/\rho\sigma^3)_p$$

No practical way has been found to model for complete similarity. Marine engineering practice is to model for wave resistance and correct for skin-friction resistance. See Sec. 11.3.

DIMENSIONAL ANALYSIS

Dimensional analysis is the mathematics of dimensions and quantities and provides procedural techniques whereby the variables that are assumed to be significant in a problem can be formed into dimensionless parameters, the number of parameters being less than the number of variables. This is a great advantage, because fewer experimental runs are then required to establish a relationship between the parameters than between the variables. While the user is not presumed to have any knowledge of the fundamental physical equations, the more knowledgeable the user, the better the results. If any significant variable or variables are omitted, the relationship obtained from dimensional analysis will not apply to the physical problem. On the other hand, inclusion of all possible variables will result in losing the principal advantage of dimensional analysis, i.e., the reduction of the amount of experimental data required to establish relationships. Two formal methods of dimensional analysis are used, the **method of Lord Rayleigh** and **Buckingham's II theorem.**

Dimensions used in mechanics are mass M, length L, time T, and force F. Corresponding **units** for these dimensions are the slug (kilogram), the foot (metre), the second (second), and the pound force (newton). Any system in mechanics can be defined by three fundamental dimensions. Two systems are used, the force (FLT) and the mass (MLT). In the force system, mass is a derived quantity and in the mass system, force is a derived quantity. Force and mass are related by Newton's law: $F = MLT^{-2}$ and $M = FL^{-1}T^2$. Table 3.3.8 shows common variables and their dimensions and units.

Lord Rayleigh's method uses algebra to determine interrelationships among variables. While this method may be used for any number of variables, it becomes relatively complex and is not generally used for more than four. This method is most easily described by example.

EXAMPLE. In laminar flow, the unit shear stress τ is some function of the fluid dynamic viscosity μ, the velocity difference dU between adjacent laminae separated by the distance dy. Develop a relationship.

1. Write a functional relationship of the variables:

$$\tau = f(\mu, dU, \mathbf{dy})$$

Assume $\tau = K(\mu^a dU^b dy^c)$.

2. Write a dimensional equation in either FLT or MLT system:

$$(FL^{-2}) = K(FL^{-2}T)^a(LT^{-1})^b(L)^c$$

3. Solve the dimensional equation for exponents:

$$\begin{array}{ll|rcl} & & \tau & & \mu \quad dU \quad dy \\ \hline \text{Force} & F & 1 & = & a + 0 + 0 \\ \text{Length} & L & -2 & = & -2a + b + c \\ \text{Time} & T & 0 & = & a - b + 0 \end{array}$$

Solution: $a = 1, b = 1, c = -1$

4. Insert exponents in the functional equation: $\tau = K(\mu^a dU^b dy^c) = K(\mu^1 du^1 dy^{-1})$, or $K = (\mu dU/\tau dy)$. This was based on the assumption of $\tau = K(\mu^a dU^b dy^c)$. The general relationship is $K = f(\mu dU/\tau dy)$. The functional relationship cannot be obtained from dimensional analysis. Only physical analysis and/or experiments can determine this. From both physical analysis and experimental data,

$$\tau = \mu \, dU/dy$$

The Buckingham II theorem serves the same purpose as the method of Lord Rayleigh for deriving equations expressing one variable in terms of its dependent variables. The II theorem is preferred when the number of variables exceeds four. Application of the II theorem results in the formation of dimensionless parameters called π ratios. These π ratios have no relation to 3.14159. . . . The II theorem will be illustrated in the following example.

EXAMPLE. Experiments are to be conducted with gas bubbles rising in a still liquid. Consider a gas bubble of diameter D rising in a liquid whose density is ρ, surface tension σ, viscosity μ, rising with a velocity of V in a gravitational field of g. Find a set of parameters for organizing experimental results.

1. List all the physical variables considered according to type: geometric, kinematic, or dynamic.

Table 3.3.8 Dimensions and Units of Common Variables

		Dimensions		Units	
Symbol	Variable	MLT	FLT	USCS*	SI
		Geometric			
L	Length	L		ft	m
A	Area	L^2		ft^2	m^2
V	Volume	L^3		ft^3	m^3
		Kinematic			
t	Time	T		s	s
ω	Angular velocity	T^{-1}		s^{-1}	s^{-1}
f	Frequency				
V	Velocity	LT^{-1}		ft/s	m/s
v	Kinematic viscosity	L^2T^{-1}		ft^2/s	m^2/s
Q	Volume flow rate	L^3T^{-1}		ft^3/s	m^3/s
α	Angular acceleration	T^{-2}		s^{-2}	s^{-2}
a	Acceleration	LT^{-2}		ft/s^{-2}	m/s^2
		Dynamic			
ρ	Density	ML^{-3}	$FL^{-4}T^2$	slug/ft^3	kg/m^3
M	Mass	M	$FL^{-1}T^2$	slugs	kg
I	Moment of inertia	ML^2	FLT^2	slug · ft^2	kg · m^2
μ	Dynamic viscosity	$ML^{-1}T^{-1}$	$FL^{-2}T$	slug/ft · s	kg/m · s
M	Mass flow rate	MT^{-1}	$FL^{-1}T^{-1}$	slug/s	kg/s
MV	Momentum	MLT^{-1}	FT	lbf · s	N · s
Ft	Impulse				
$M\omega$	Angular momentum	ML^2T^{-1}	FLT	slug · ft^2/s	kg · m^2/s
γ	Specific weight	$ML^{-2}T^{-2}$	FL^{-3}	lbf/ft^3	N/m^3
p	Pressure	$ML^{-1}T^{-2}$	FL^{-2}	lbf/ft^2	N/m^2
τ	Unit shear stress				
E	Modulus of elasticity				
σ	Surface tension	MT^{-2}	FL^{-1}	lbf/ft	N/m
F	Force	MLT^{-2}	F	lbf	N
E	Energy	ML^2T^{-2}	FL	lbf · ft	J
W	Work				
FL	Torque				
P	Power	ML^2T^{-3}	FLT^{-1}	lbf · ft/s	W
v	Specific volume	$M^{-1}L^3$	$F^{-1}L^4T^{-2}$	ft^3/lbm	m^3/kg

*United States Customary System.

2. Choose either the *FLT* or *MLT* system of dimensions.
3. Select a ''basic group'' of variables characteristic of the flow as follows:
   - *a.* B_G, a geometric variable
   - *b.* B_K, a kinematic variable
   - *c.* B_D, a dynamic variable (if three dimensions are used)
4. Assign A numbers to the remaining variables starting with A_1.

Type	*Symbol*	*Description*	*Dimensions*	*Number*
Geometric	D	Bubble diameter	L	B_G
Kinematic	V	Bubble velocity	LT^{-1}	B_K
	g	Acceleration of gravity	LT^{-2}	A_1
Dynamic	ρ	Liquid density	ML^{-3}	B_D
	σ	Surface tension	MT^{-2}	A_2
	μ	Liquid viscosity	$ML^{-1}T^{-1}$	A_3

5. Write the basic equation for each π ratio as follows:

$$\pi_1 = (B_G)^{x1}(B_K)^{y1}(B_D)^{z1}(A_1)$$
$$\pi_2 = (B_G)^{x2}(B_K)^{y2}(B_D)^{z2}(A_2) \quad . . . \quad \pi_n = (B_G)^{xn}(B_K)^{yn}(B_D)^{zn}(A_n)$$

Note that the number of π ratios is equal to the number of A numbers and thus equal to the number of variables less the number of fundamental dimensions in a problem.

6. Write the dimensional equations and use the algebraic method to determine the value of exponents x, y, and z for each π ratio. Note that for all π ratios, the sum of the exponents of a given dimension is zero.

$$\pi_1 = (B_G)^{x1}(B_K)^{y1}(B_D)^{z1}(A_1) = (D)^{x1}(V)^{y1}(\rho)^{z1}(g)$$
$$(M^0L^0T^0) = (L^{x1})(L^{y1}T^{-y1})(M^{z1}L^{-3z1})(LT^{-2})$$

Solution: $x_1 = 1, y_1 = -2, z_1 = 0$

$$\pi_1 = D^1V^{-2}\rho^0g = Dg/V^2$$
$$\pi_2 = (B_G)^{x2}(B_K)^{y2}(B_D)^{z2}(A_2) = (D)^{x2}(V)^{y2}(\rho)^{z2}(\sigma)$$
$$(M^0L^0T^0) = (L^{x2})(L^{y2}T^{-y2})(M^{z2}L^{-3z2})(MT^{-2})$$

Solution: $x_2 = -1, y_2 = -2, z_2 = -1$

$$\pi_2 = D^{-1}V^{-2}\rho^{-1}\sigma = \sigma/DV^2\rho$$
$$\pi_3 = (B_G)^{x3}(B_K)^{y3}(B_D)^{z3}(A_3) = (D)^{x3}(V)^{y3}(\rho)^{z3}(\mu)$$
$$(M^0L^0T^0) = (L^{x3})(L^{y3}T^{-y3})(M^{z3}L^{-3z3})(ML^{-1}T^{-1})$$

Solution: $x_3 = -1, y_3 = -1, z_3 = -1$

$$\pi_3 = D^{-1}V^{-1}\rho^{-1}\mu = \mu/DV\rho$$

7. Convert π ratios to conventional practice. One statement of the Buckingham II theorem is that any π ratio may be taken as a function of all the others, or $f(\pi_1, \pi_2, \pi_3, \ldots, \pi_n) = 0$. This equation is mathematical shorthand for a functional statement. It could be written, for example, as $\pi_2 = f(\pi_1, \pi_3, \ldots, \pi_n)$. This equation states that π_2 is some function of π_1 and π_3 through π_n but is not a statement of *what* function π_2 is of the other π ratios. This can be determined only by physical and/or experimental analysis. Thus we are free to substitute *any* function in the equation; for example, π_1 may be replaced with $2\pi_1^{-1}$ or π_n with $a\pi_n^b$.

The procedures set forth in this example are designed to produce π ratios containing the same terms as those resulting from the application of the principles of similarity so that the physical significance may be understood. However, any other combinations might have been used. The only real requirement for a ''basic group'' is that it contain the same number of terms as there are dimensions in a problem and that each of these dimensions be represented in it.

The π ratios derived for this example may be converted into conventional practice as follows:

$$\pi_1 = Dg/V^2$$

is recognized as the inverse of the square root of the Froude number **F**

$$\pi_2 = \sigma/DV^2\rho$$

is the inverse of the Weber number **W**

$$\pi = \mu/DV\rho$$

is the inverse of the Reynolds number **R**

Let $\pi_1 = f(\pi_2, \pi_3)$
Then $V = K(Dg)^{1 \div 2}$
where $K = f(\mathbf{W}, \mathbf{R})$

This agrees with the results of the dynamic-similarity analysis of liquid surfaces. This also permits a reduction in the experimental program from variations of six variables to three dimensionless parameters.

FORCES OF IMMERSED OBJECTS

Drag and Lift When a fluid impinges on an object as shown in Fig. 3.3.20, the undisturbed fluid pressure p and the velocity V change. Writing Bernoulli's equation for two points on the surface of the object, the point S being the most forward point and point A being any other point, we have, for horizontal flow,

$$p + \rho V^2/2 = p_S + \rho V_S^2/2 = p_A + \rho V_A^2/2$$

At point S, $V_S = 0$, so that $p_S = p + \rho V^2/2$. This is called the **stagnation** point, and p_S is the **stagnation pressure.** Since point A is any other point, the result of the fluid impingement is to create a pressure $p_A = p + \rho(V^2 - V_A^2)/2$ acting normal to every point on the surface of the object.

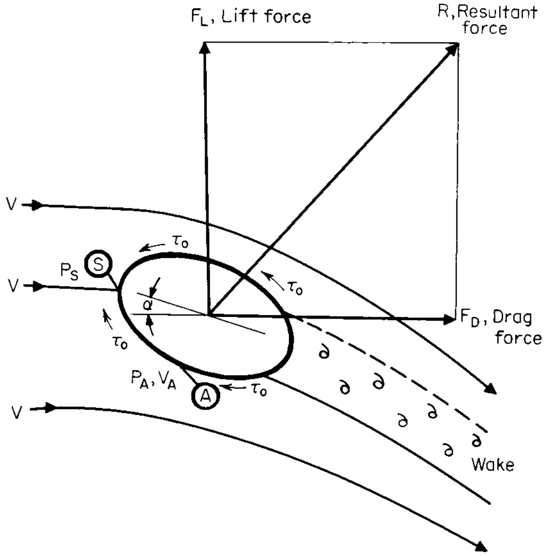


Fig. 3.3.20 Notation for drag and lift.

In addition, a frictional force $F_f = \tau_0 A_s$ tangential to the surface area A_s opposes the motion. The sum of these forces gives the resultant force R acting on the body. The resultant force R is resolved into the **drag** component F_D parallel to the flow and **lift** component F_L perpendicular to the fluid motion. Depending upon the shape of the object, a wake may be formed which sheds eddies with a frequency of f. The angle α is called the angle of attack. (See Secs. 11.4 and 11.5.)

From dimensional analysis or dynamic similarity,

$$f(\mathbf{C}_p, \mathbf{R}, \mathbf{M}, \mathbf{S}) = 0$$

The formation of a wake depends upon the Reynolds number, or $\mathbf{S} = f(\mathbf{R})$. This reduces the functional relation to $f(\mathbf{C}_p, \mathbf{R}, \mathbf{M}) = 0$. Since the drag and lift forces may be considered independently,

$$F_D = C_D \rho V^2(A)/2$$

where $C_D = f(\mathbf{R}, \mathbf{M})$, and A = characteristic area.

$$F_L = C_L \rho V^2(A)/2$$

where $C_L = f(\mathbf{R}, \mathbf{M})$.

It is evident from Fig. 3.3.20 that C_D and C_L are also functions of the angle of attack. Since the drag force arises from two sources, the pressure or shape drag F_p and the skin-friction drag F_f due to wall shear stress τ_0, the drag coefficient is made up of two parts:

$$F_D = F_p + F_f = C_D \rho A V^2/2 = C_p \rho A V^2/2 + C_f \rho A_s V^2/2$$

or $$C_D = C_p + C_f A_s/A$$

where C_p is the coefficient of pressure, C_f the skin-friction coefficient, and A_s the characteristic area for shear.

Skin-Friction Drag Figure 3.3.21 shows a fluid approaching a smooth flat plate with a uniform velocity profile of V. As the fluid passes over the plate, the velocity at the plate surface is zero and increases to V at some distance δ from the surface. The region in which the velocity varies from 0 to V is called the **boundary layer.** For some distance along the plate, the flow within the boundary layer is laminar, with viscous forces predominating, but in the transition zone as the inertia forces become larger, a turbulent layer begins to form and increases as the laminar layer decreases.

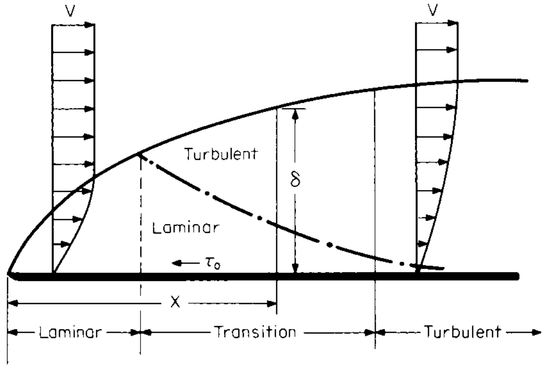


Fig. 3.3.21 Boundary layer along a smooth flat plate.

Boundary-layer thickness and skin-friction drag for incompressible flow over smooth flat plates may be calculated from the following equations, where $\mathbf{R}_X = \rho VX/\mu$:

Laminar

$$\delta/X = 5.20\ \mathbf{R}_X^{-1/2} \qquad 0 < \mathbf{R}_X < 5 \times 10^5$$
$$C_f = 1.328\ \mathbf{R}_X^{-1/2} \qquad 0 < \mathbf{R}_X < 5 \times 10^5$$

Turbulent

$$\delta/X = 0.377\ \mathbf{R}_X^{-1/5} \qquad 5 \times 10^4 < \mathbf{R}_X < 10^6$$
$$\delta/X = 0.220\ \mathbf{R}_X^{-1/6} \qquad 10^6 < \mathbf{R}_X < 5 \times 10^8$$
$$C_f = 0.0735\ \mathbf{R}_X^{-1/5} \qquad 2 \times 10^5 < \mathbf{R}_X < 10^7$$
$$C_f = 0.455\ (\log_{10} \mathrm{R}_X)^{-2.58} \qquad 10^7 < \mathbf{R}_X < 10^8$$
$$C_f = 0.05863\ (\log_{10} C_f R_X)^{-2} \qquad 10^8 < \mathbf{R}_x < 10^9$$

Transition The Reynolds number at which the boundary layer changes depends upon the roughness of the plate and degree of turbulence. The generally accepted number is 500,000, but the transition can take place at Reynolds numbers higher or lower. (Refer to Secs. 11.4 and 11.5.) For transition at any Reynolds number $\mathbf{R}_X$,

$$C_f = 0.455\ (\log_{10} \mathbf{R}_X)^{-2.58} - (0.0735\ \mathbf{R}_t^{4/5} - 1.328\ 8\ \mathbf{R}_t^{1/2})\ \mathbf{R}_X^{-1}$$

For $\mathbf{R}_t = 5 \times 10^5$, $C_f = 0.455\ (\log_{10} \mathbf{R}_X)^{-2.58} - 1{,}725\ \mathbf{R}_X^{-1}$.

Pressure Drag Experiments with sharp-edged objects placed perpendicular to the flow stream indicate that their drag coefficients are essentially constant at Reynolds numbers over 1,000. This means that the drag for $\mathbf{R}_X > 10^3$ is pressure drag. Values of C_D for various shapes are given in Sec. 11 along with the effects of Mach number.

Wake Frequency An object in a fluid stream may be subject to the downstream periodic shedding of vortices from first one side and then the other. The frequency of the resulting transverse (lift) force is a function of the stream Strouhal number. As the wake frequency approaches the natural frequency of the structure, the periodic lift force increases asymptotically in magnitude, and when resonance occurs, the structure fails. Neglecting to take this phenomenon into account in design has been responsible for failures of electric transmission lines, submarine periscopes, smokestacks, bridges, and thermometer wells. The wake-frequency characteristics of **cylinders** are shown in Fig. 3.3.22. At a Reynolds number of about 20, vortices begin to shed alternately. Behind the cylinder is a staggered stable arrangement of vortices known as the ''Kármán vortex trail.'' At a Reynolds number of about 10^5, the flow changes from laminar to turbulent. At the end of the transition zone ($\mathbf{R} \approx 3.5 \times 10^5$), the flow becomes turbulent, the alternate shedding stops, and the wake is aperiodic. At the end of the supercritical zone ($\mathbf{R} \approx 3.5 \times 10^6$), the wake continues to be turbulent, but the shedding again becomes alternate and periodic.

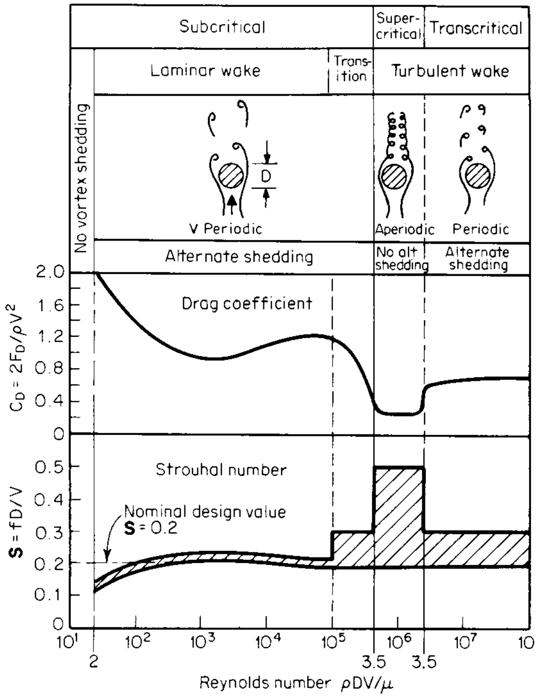


Fig. 3.3.22 Flow around a cylinder. *(From Murdock, ''Fluid Mechanics and Its Applications,'' Houghton Mifflin, 1976.)*

The alternating **lift force** is given by

$$F_L(t) = C_L \rho V^2 A \sin (2\pi ft)/2$$

where t is the time. For an analysis of this force in the subcritical zone, see Belvins (Murdock, ''Fluid Mechanics and Its Applications,'' Houghton Mifflin, 1976). For design of steel stacks, Staley and Graven (ASME 72PET/30) recommend $C_L = 0.8$ for $10^4 < \mathbf{R} < 10^5$, $C_L = 2.8 - 0.4 \log_{10} \mathbf{R}$ for $\mathbf{R} = 10^5$ to 10^6, and $C_L = 0.4$ for $10^6 < \mathbf{R} < 10^7$.

The **Strouhal number** is nearly constant to $\mathbf{R} = 10^5$, and a nominal design value of 0.2 is generally used. Above $\mathbf{R} = 10^5$, data from different experimenters vary widely, as indicated by the crosshatched zone of Fig. 3.3.22. This wide zone is due to experimental and/or measurement difficulties and the dependence on surface roughness to ''trigger'' the boundary layer. Examination of Fig. 3.3.22 indicates an inverse relation of Strouhal number to drag coefficient.

Observation of actual structures shows that they vibrate at their natural frequency and with a mode shape associated with their fundamental (first) mode during vortex excitation. Based on observations of actual stacks and wind-tunnel tests, Staley and Graven recommend a constant Strouhal number of 0.2 for all ranges of Reynolds number. The ASME recommends $\mathbf{S} = 0.22$ for thermowell design (''Temperature Measurement,'' PTC 19.3). Until such time as the value of the Strouhal number above $\mathbf{R} = 10^5$ has been firmly established, designers of structures in this area should proceed with caution.

FLOW IN PIPES

Parameters for Pipe Flow The forces acting on a fluid flowing through and completely filling a horizontal pipe are inertia, viscous, pressure, and elastic. If the surface roughness of the pipe is ε, either similarity or dimensional analysis leads to $\mathbf{C}_p = f(\mathbf{R}, \mathbf{M}, L/D, \varepsilon/D)$, which may be written for incompressible fluids as $\Delta p = \mathbf{C}_p V^2/2 = K\rho V^2/2$, where K is the **resistance coefficient** and ε/D the **relative roughness** of the pipe surface, and the resistance coefficient $K = f(\mathbf{R}, L/D, \varepsilon/D)$. The pressure loss may be converted to the terms of **lost head:** $h_f = \Delta p/\gamma = KV^2/2g$. Conventional practice is to use the **friction factor** f, defined as $f = KD/L$ or $h_f = KV^2/2g = (fL/D)V^2/2g$, where $f = f(\mathbf{R}, \varepsilon/D)$. When a fluid flows into a pipe, the boundary layer starts at the entrance, as shown in Fig. 3.3.23, and grows continuously until it fills the pipe. From the equation of motion $dh_f = \tau\, dL/\gamma R_h$ and for circular ducts $R_h = D/4$. Comparing wall shear stress τ_0 with friction factor results in the following: $\tau_0 = f\rho V^2/8$.

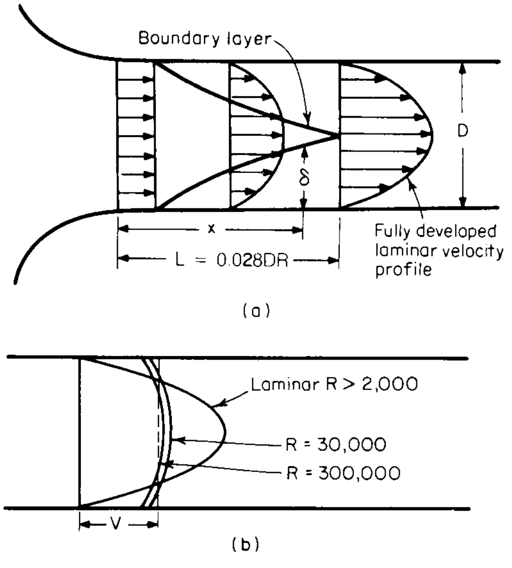


Fig. 3.3.23 Velocity profiles in pipes.

Laminar Flow In this type of flow, the resistance is due to viscous forces only so that it is independent of the pipe surface roughness, or $\tau_0 = \mu\, dU/dy$. Application of this equation to the equation of motion and the friction factor yields $f = 64/\mathbf{R}$. Experiments show that it is possible to maintain laminar flow to very high Reynolds numbers if care is taken to increase the flow gradually, but normally the slightest disturbance will destroy the laminar boundary layer if the value of Reynolds number is greater than 4,000. In a like manner, flow initially turbulent

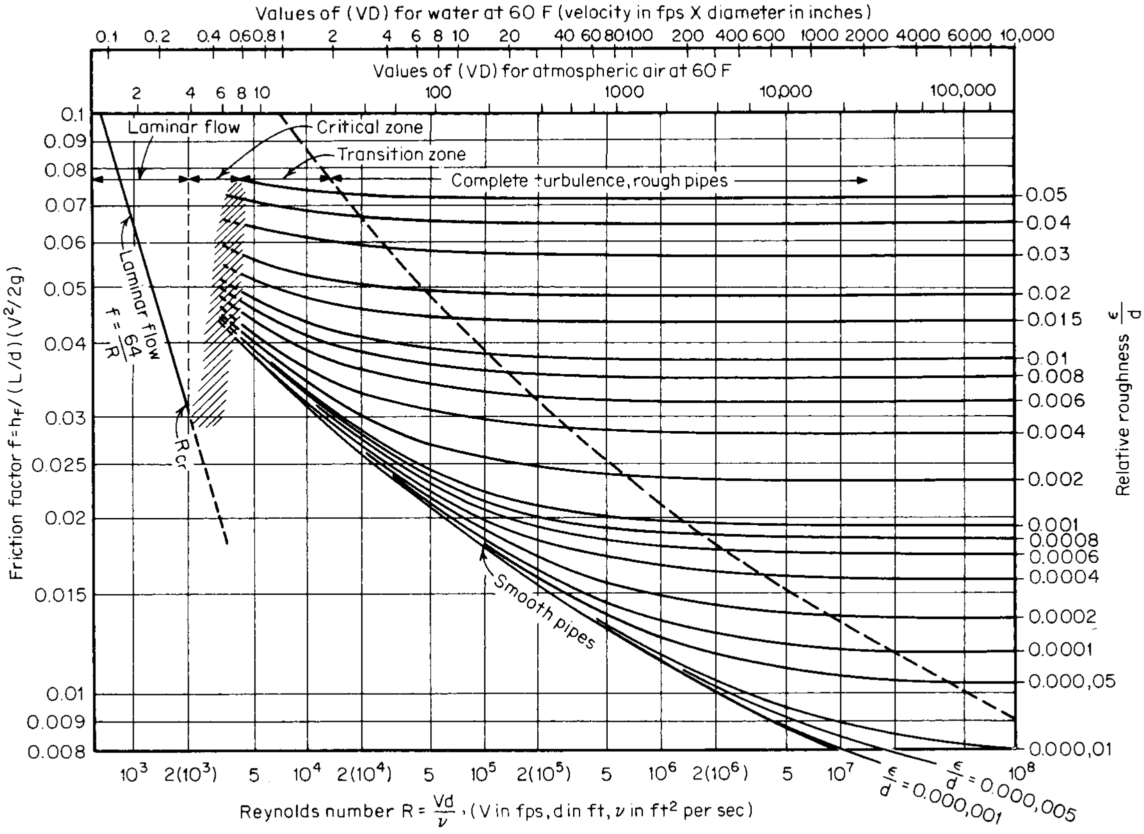


Fig. 3.3.24 Friction factors for flow in pipes.

can be maintained with care to very low Reynolds numbers, but the slightest upset will result in laminar flow if the Reynolds number is less than 2,000. The Reynolds-number range between 2,000 and 4,000 is called the **critical zone** (Fig. 3.3.24). Flow in the zone is **unstable,** and designers of piping systems must take this into account.

EXAMPLE. Glycerin at 68°F (20°C) flows through a horizontal pipe 1 in in diameter and 20 ft long at a rate of 0.090 lbm/s. What is the pressure loss? From the continuity equation $V = Q/A = (m/\rho g)/(\pi D^2/4) = [0.090/(2.447 \times 32.17)]/[(\pi/4)(1/12)^2] = 0.2096$ ft/s. The Reynolds number $\mathbf{R} = \rho VD/\mu = (2.447)(0.2096)(1/12)/(29{,}500 \times 10^{-6}) = 1.449$. $\mathbf{R} < 2{,}000$; therefore, flow is laminar and $f = 64/\mathbf{R} = 64/1.449 = 44.17$. $K = fL/D = 44.17 \times 20(1/12) = 10{,}600$. $\Delta p = K\rho V^2/2 = 10{,}600 \times 2.447\,(0.2096)^2/2 = 569.8$ lbf/ft^2 = 569.8/144 = 3.957 lbf/in^2 (2.728×10^4 N/m^2).

Turbulent Flow The friction factor for Reynolds number over 4,000 is computed using the Colebrook equation:

$$\frac{1}{\sqrt{f}} = -2 \log_{10} \left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathbf{R}\sqrt{f}} \right)$$

Figure 3.3.24 is a graphical presentation of this equation (Moody, *Trans. ASME,* 1944, pp. 671–684). Examination of the Colebrook equation indicates that if the value of surface roughness ε is small compared with the pipe diameter ($\varepsilon/D \rightarrow 0$), the friction factor is a function of Reynolds number only. A **smooth pipe** is one in which the ratio $(\varepsilon/D)/3.7$ is small compared with $2.51/\mathbf{R}\sqrt{f}$. On the other hand, as the Reynolds number increases so that $2.51/\mathbf{R}\sqrt{f} \rightarrow 0$, the friction factor becomes a function of relative roughness only and the pipe is called a **rough pipe.** Thus the same pipe may be smooth under one flow condition, and rough under another. The reason for this is that as the Reynolds number increases, the thickness of the laminar sublayer decreases as shown in Fig. 3.3.21, exposing the surface roughness to flow. Values of absolute roughness ε are given in Table 3.3.9. The variation of friction factor shown in Fig. 3.3.9 is for new, clean pipes. The change of friction factor with **age** depends upon the chemical properties of the fluid and the piping material. Published data for flow of water through wrought-iron or cast-iron pipes show as much as 20 percent increase after a few months to 500 percent after 20 years. When necessary to allow for service life, a study of specific conditions is recommended. The calculation of friction factor to four significant figures in the examples to follow is only for numerical comparison and should not be construed to mean accuracy.

Table 3.3.9 Values of Absolute Roughness, New Clean Commercial Pipes

Type of pipe or tubing	ε ft (0.3048 m) $\times 10^{-6}$ Range	Design	Probable max variation of f from design, %
Asphalted cast iron	400	400	−5 to +5
Brass and copper	5	5	−5 to +5
Concrete	1,000 10,000	4,000	−35 to 50
Cast iron	850	850	−10 to +15
Galvanized iron	500	500	0 to +10
Wrought iron	150	150	−5 to 10
Steel	150	150	−5 to 10
Riveted steel	3,000 30,000	6,000	−25 to 75
Wood stave	600 3,000	2,000	−35 to 20

SOURCE: Compiled from data given in ''Pipe Friction Manual,'' Hydraulic Institute, 3d ed., 1961.

Engineering Calculations Engineering pipe computations usually fall into one of the following classes:

1. Determine pressure loss Δp when Q, L, and D are known.
2. Determine flow rate Q when L, D, and Δp are known.
3. Determine pipe diameter D when Q, L, and Δp are known.

Pressure-loss computations may be made to engineering accuracy using an expanded version of Fig. 3.3.24. Greater precision may be obtained by using a combination of Table 3.3.9 and the Colebrook equation, as will be shown in the example to follow. Flow rate may be determined by direct solution of the Colebrook equation. Computation of pipe diameter necessitates the trial-and-error method of solution.

EXAMPLE. *Case 1:* 2,000 gal/min of 68°F (20°C) water flow through 500 ft of cast-iron pipe having an internal diameter of 10 in. At point 1 the pressure is 10 lbf/in² and the elevation 150 ft, and at point 2 the elevation is 100 ft. Find p_2.

From continuity $V = Q/A = [2{,}000 \times (231/1{,}728)/60]/[(\pi/4)(10/12)^2] = 8.170$ ft/s. Reynolds number $\mathbf{R} = \rho VD/\mu = (1.937)(8.170)(10/12)/(20.92 \times 10^{-6}) = 6.304 \times 10^5$. $\mathbf{R} > 4{,}000$ $\therefore$ flow is turbulent. $\varepsilon/D = (850 \times 10^{-6})/(10/12) = 1.020 \times 10^{-3}$.

Determine f: from Fig. 3.3.24 by interpolation $f = 0.02$. Substituting this value on the right-hand side of the Colebrook equation,

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathbf{R}\sqrt{f}}\right)$$

$$= -2\log_{10}\left[\frac{1.020 \times 10^{-3}}{3.7} + \frac{2.51}{(6.305 \times 10^5)\sqrt{0.02}}\right]$$

$$\frac{1}{\sqrt{f}} = 7.035 \qquad f = 0.02021$$

$$\text{Resistance coefficient } K = \frac{fL}{D} = \frac{0.02021 \times 500}{10/12}$$

$$\begin{aligned} K &= 12.13 \\ h_{1f2} &= KV^2/2g = 12.13 \times (8.170)^2/2 \times 32.17 \\ h_{1f2} &= 12.58 \text{ ft} \end{aligned}$$

Equation of motion: $p_1/\gamma + V_1^2/2g + z_1 = p_2/\gamma + V_2^2/2g + z_2 + h_{1f2}$. Noting that $V_1 = V_2 = V$ and solving for p_2,

$$\begin{aligned} p_2 &= p_1 + \gamma(z_1 - z_2 - h_{1f2}) \\ &= 144 \times 10 + (1.937 \times 32.17)(150 - 100 - 12.58) \\ p_2 &= 3{,}772 \text{ lbf/ft}^2 = 3{,}772/144 = 26.20 \text{ lbf/in}^2\ (1.806 \times 10^5 \text{ N/m}^2) \end{aligned}$$

EXAMPLE. *Case 2:* Gasoline (sp. gr. 0.68) at 68°F (20°C) flows through a 6-in schedule 40 (ID = 0.5054 ft) welded steel pipe with a head loss of 10 ft in 500 ft. Determine the flow. This problem may be solved directly by deriving equations that do not contain the flow rate Q.

$$\text{From } h_f = \left(\frac{fL}{D}\right)\frac{V^2}{2g}, \qquad V = (2gh_fD)^{1/2}(fL)^{1/2}$$

$$\text{From } \mathbf{R} = \rho VD/\mu, \qquad V = \mathbf{R}\mu/\rho D$$

Equating the above and solving,

$$\begin{aligned} \mathbf{R}\sqrt{f} &= (\rho D/\mu)(2gh_fD/L)^{1/2} \\ &= (1.310 \times 0.5054/5.98 \times 10^{-6}) \\ &\quad \times (2 \times 32.17 \times 10 \times 0.5054/500)^{1/2} = 89{,}285 \end{aligned}$$

which is now in a form that may be used directly in the Colebrook equation:

$$\varepsilon/D = 150 \times 10^{-6}/0.5054 = 2.968 \times 10^{-4}$$

From the Colebrook equation,

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathbf{R}\sqrt{f}}\right)$$

$$= -2\log_{10}\left(\frac{2.968 \times 10^{-4}}{3.7} + \frac{2.51}{89{,}285}\right)$$

$$\frac{1}{\sqrt{f}} = 7.931 \qquad f = 0.01590$$

$$\begin{aligned} \mathbf{R} &= 89{,}285/\sqrt{f} = 89{,}285 \times 7.93 = 7.08 \times 10^5 \\ \mathbf{R} &> 4{,}000 \therefore \text{flow is turbulent} \\ V &= \mathbf{R}\mu/\rho D = (7.08 \times 10^5 \times 5.98 \\ &\quad \times 10^{-6})/(1.310 \times 0.5054) = 6.396 \text{ ft/s} \\ Q &= AV = (\pi/4)(0.5054)^2(6.396) \\ Q &= 1.283 \text{ ft}^3\text{/s}\ (3.633 \times 10^{-2} \text{ m}^3\text{/s}^1) \end{aligned}$$

EXAMPLE. *Case 3;* Water at 68°F (20°C) is to flow at a rate of 500 ft³/s through a concrete pipe 5,000 ft long with a head loss not to exceed 50 ft. Determine the diameter of the pipe. This problem may be solved by trial and error using methods of the preceding example. First trial: Assume any diameter (say 1 ft).

$$\begin{aligned} \mathbf{R}\sqrt{f} &= (\rho D/\mu)(2gh_fD/L)^{1/2} \\ &= (1.937D/20.92 \times 10^{-6}) \times (2 \times 32.17 \times 50D/5{,}000)^{1/2} \\ &= 74{,}269D^{3/2} = 74{,}269(1)^{3/2} = 74{,}269 \\ \varepsilon/D_1 &= 4{,}000 \times 10^{-6}/D = 4{,}000 \times 10^{-6}/(1) = 4{,}000 \times 10^{-6} \end{aligned}$$

$$\frac{1}{\sqrt{f_1}} = -2\log_{10}\left(\frac{\varepsilon/D_1}{3.7} + \frac{2.51}{\mathbf{R}\sqrt{f_1}}\right)$$

$$= -2\log_{10}\left(\frac{4{,}000 \times 10^{-6}}{3.7} + \frac{2.51}{74{,}269}\right)$$

$$\frac{1}{\sqrt{f_1}} = 5.906 \qquad f_1 = 0.02867$$

$$\begin{aligned} \mathbf{R}_1 &= 74{,}269/\sqrt{f_1} = 74{,}269 \times 5.906 = 438{,}600 \\ V_1 &= \mathbf{R}\mu/\rho D_1 = (438{,}600 = 20.92 \times 10^{-6})/(1.937 \times 1) \\ V_1 &= 4.737 \text{ ft/s} \\ Q_1 &= A_1V_1 = [\pi(1)^2/4]4.737 = 3.720 \text{ ft}^3\text{/s} \end{aligned}$$

For the same loss and friction factor,

$$D_2 = D_1(Q/Q_1)^{2/5} = (1)(500/3.720)^{2/5} = 7.102 \text{ ft}$$

For the second trial use $D_2 = 7.102$, which results in $Q = 502.2$ ft³/s. Since the nearest standard size would be used, additional trials are unnecessary.

Velocity Profile Figure 3.3.23*a* shows the formation of a laminar velocity profile. As the fluid enters the pipe, the boundary layer starts at the entrance and grows continuously until it fills the pipe. The flow while the boundary is growing is called **generating flow.** When the boundary layer completely fills the pipe, the flow is called **established flow.** The distance required for establishing **laminar flow** is $L/D \approx 0.028$ **R.** For **turbulent flow,** the distance is much shorter because of the turbulence and not dependent upon Reynolds number, L/D being from 25 to 50.

Examination of Fig. 3.3.23*b* indicates that as the Reynolds number increases, the velocity distribution becomes ‘‘flatter’’ and the flow approaches **one-dimensional.** The velocity profile for laminar flow is parabolic, $U/V = 2[1 - (r/r_o)^2]$ and for **turbulent flow, logarithmic** (except for the very thin laminar boundary layer), $U/V = 1 + 1.43\sqrt{f} + 2.15\sqrt{f}\log_{10}(1 - r/r_o)$. The use of the average velocity produces an error in the computation of kinetic energy. If α is the **kinetic-energy correction factor,** the true kinetic-energy change per unit mass between two points on a flow system $\Delta KE = \alpha_1V_1^2/2g_c - \alpha_2V_2^2/2g_c$, where $\alpha = (1/AV^3)\int U^3dA$. For **laminar flow,** $\alpha = 2$ and for **turbulent flow,** $\alpha \approx 1 + 2.7f$. Of interest is the **pipe factor** V/U_{max}; for **laminar flow,** $V/U_{max} = 1/2$ and for **turbulent flow,** $V/U_{max} = 1 + 1.43\sqrt{f}$. The location at which the local velocity equals the average velocity for **laminar flow** is $U = V$ at $r/r_o = 0.7071$ and for **turbulent flow** is $U = V$ at $r/r_o = 0.7838$.

Compressible Flow At the present time, there are no true analytical solutions for the computation of actual characteristics of compressible fluids flowing in pipes. In the real flow of a compressible fluid in a pipe, the amount of heat transferred and its direction are dependent upon the amount of insulation, the temperature gradient between the fluid and ambient temperatures, and the heat-transfer coefficient. Each condition requires an individual application of the principles of thermodynamics and heat transfer for its solution.

Conventional engineering practice is to use one of the following methods for flow computation.

1. Assume **adiabatic** flow. This approximates the flow of compressible fluids in short, insulated pipelines.
2. Assume **isothermal** flow. This approximates the flow of gases in long, uninsulated pipelines where the fluid and ambient temperatures are nearly equal.

Adiabatic Flow If the Mach number is less than ¼, results within normal engineering-accuracy requirements may be obtained by considering the fluid to be incompressible. A detailed discussion of and methods for the solution of compressible adiabatic flow are beyond the scope of this section, and any standard gas-dynamics text should be consulted.

Isothermal Flow The equation of motion for a horizontal piping system may be written as follows:

$$dp + \rho V\,dV + \gamma\,dh_f = 0$$

noting, from the continuity equation, that $\rho V = \dot{M}/A = G$, where G is the **mass velocity** in slugs/(ft^2)(s), and that $\gamma\,dh_f = [(f/D)\rho V^2/2]dL = [(f/D)GV/2]dL$. Substituting in the above equation of motion and dividing by $GV/2$ results in

$$\frac{2\rho\,dp}{G^2} + \frac{2dV}{V} + \left(\frac{f}{D}\right) dL = 0$$

Integrating for an isothermal process ($p/\rho = C$) and assuming f is a constant,

$$\frac{\rho_1 p_1}{G^2}\left[\left(\frac{p_2}{p_1}\right)^2 - 1\right] + 2\log_e\left(\frac{V_2}{V_1}\right) + \frac{fL}{D} = 0$$

Noting that $A_1 = A_2$, $V_2/V_1 = \rho_1/\rho_2 = p_1/p_2$, and solving for G,

$$G = \left\{\frac{\rho_1 p_1[1 - (p_2/p_1)^2]}{2\log_e(p_1/p_2) + fL/D}\right\}^{1/2}$$

The Reynolds number may be written as

$$\mathbf{R} = \frac{\rho VD}{\mu} = \frac{GD}{\mu} \qquad \text{and} \qquad G = \mathbf{R}\frac{\mu}{D}$$

The value of $\mathbf{R}\sqrt{f}$ may be obtained from the simultaneous solution of the two equations for G, assuming that $2\log_e p_1/p_2$ is small compared with fL/D.

$$\mathbf{R}\sqrt{f} \approx \left\{\frac{D^3\rho_1 p_1}{\mu^2 L}\left[1 - \left(\frac{p_2}{p_1}\right)\right]\right\}^{1/2}$$

EXAMPLE. Air at 68°F (20°C) is flowing isothermally through a horizontal straight standard 1-in steel pipe (inside diameter = 1.049 in). The pipe is 200 ft long, the pressure at the pipe inlet is 74.7 lbf/in^2, and the pressure drop through the pipe is 5 lbf/in^2. Find the flow rate in lbm/s. From the equation of state $\rho_1 = p/g_cRT = (144 \times 74.7)/(32.17 \times 53.34 \times 527.7) = 0.01188$ slugs/ft^3.

$$\begin{aligned}\mathbf{R}\sqrt{f} &= \{[(D^3\rho_1 p_1/\mu^2 L)][1 - (p_2/p_1)^2]\}^{1/2} = \{[(1.049/12)^3(0.01188)\\ &\quad \times (144 \times 74.7)/(39.16 \times 10^{-8})^2(200)[1 - (69.7/74.7)^2]\}^{1/2}\\ &= 18{,}977\end{aligned}$$

For steel pipe $\varepsilon = 150 \times 10^{-6}$ ft, $\varepsilon/D = (150 \times 10^{-6})/(1.049/12) = 1.716 \times 10^{-3}$. From the Colebrook equation,

$$\begin{aligned}\frac{1}{\sqrt{f}} &= -2\log_{10}\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathbf{R}\sqrt{f}}\right)\\ &= 2\log_{10}[(1.716 \times 10^{-3}/3.7) + (2.51)/(18{,}977)] = 6.449\\ f &= 0.02404\end{aligned}$$

$$\begin{aligned}\mathbf{R} &= (\mathbf{R}\sqrt{f})(1/\sqrt{f}) = (18{,}953)(6.449) = 122{,}200\\ \mathbf{R} &> 4{,}000 \therefore \text{flow is turbulent}\end{aligned}$$

$$\begin{aligned}G &= \left\{\frac{\rho_1 p_1[1 - (p_2/p_1)^2]}{2\log_e(p_1/p_2) + fL/D}\right\}^{1/2}\\ &= \left\{\frac{(0.01188)(144 \times 74.7)[1 - (69.7/74.7)^2]}{2\log_e(74.7/69.7) + (0.02404)(200)/(1.049/12)}\right\}^{1/2}\\ &= 0.5476 \text{ slug/(ft}^2\text{)(s)}\\ \dot{m} &= g_cAG = (32.17)(\pi/4)(1.049/12)^2(0.5476)\\ \dot{m} &= 0.1057 \text{ lbm/s } (47.94 \times 10^{-3} \text{ kg/s})\end{aligned}$$

Noncircular Pipes For the flow of fluids in noncircular pipes, the **hydraulic diameter** D_h is used. From the definition of hydraulic radius, the diameter of a circular pipe was shown to be four times its **hydraulic radius;** thus $D_h = 4R_h$. The Reynolds number thus may be written as $\mathbf{R} = \rho VD_h/\mu = GD_h/\mu$, the relative roughness as ε/D_h, and the resistance coefficient $K = fL/D_h$. With the above modifications, flows through noncircular pipes may be computed in the same manner as for circular pipes.

EXAMPLE. Air at 68°F (20°C) and 100 lbf/in^2 enters a rectangular duct 1 by 3 ft at a rate of 720 lbm/s. The duct is horizontal, 100 ft long, and made of galvanized iron. Assuming isothermal flow, estimate the pressure loss due to friction in this line. From the equation of state, $\rho_1 = p_1/g_c\text{RT}_1 = (144 \times 100)/(32.17)(53.34)(527.7) = 0.01590$ slug/ft^3. From Table 3.3.6, $R_h = bD/2(b + D) = 3 \times 1/2(3 + 1) = 0.375$ ft, and $D_h = 4R_h = 4 \times 0.375 = 1.5$ ft. For galvanized iron, $\varepsilon/D_h = 500 \times 10^{-6}/1.5 = 3.333 \times 10^{-4}$

$$\begin{aligned}G &= (\dot{m}/g_c)/A = (720/32.17)/(1 \times 3) = 7.460 \text{ slugs/(ft}^2\text{)(s)}\\ \mathbf{R} &= GD_h/\mu = (7.460)(1.5)/(39.16 \times 10^{-8})\\ &= 28{,}580{,}000 > 4{,}000 \therefore \text{flow is turbulent}\end{aligned}$$

From Fig. 3.3.24, $f \approx 0.015$

$$\begin{aligned}\frac{1}{\sqrt{f}} &= -2\log_{10}\left(\frac{\varepsilon/D_h}{3.7} + \frac{2.51}{\mathbf{R}\sqrt{f}}\right)\\ &= -2\log_{10}\left(\frac{3.333 \times 10^{-4}}{3.7} + \frac{2.51}{28{,}580{,}000\sqrt{0.015}}\right)\\ f &= 0.01530\end{aligned}$$

Solving the isothermal equation for p_2/p_1,

$$\frac{p_2}{p_1} = \left\{1 - \left(\frac{G^2}{\rho_1 p_1}\right)\left[2\log_e\left(\frac{p_1}{p_2}\right) + \frac{fL}{D_h}\right]\right\}^{1/2}$$

For first trial, assume $2\log_e(p_1/p_2)$ is small compared with fL/D:

$$\begin{aligned}p_2/p_1 &= \{1 - [(7.460)^2/(0.01590)(144 \times 100)][0 + (0.01530)(100)/1.5]\}^{1/2}\\ &= 0.8672\end{aligned}$$

Second trial using first-trial values results in 0.8263. Subsequent trials result in a balance at $p_2/p_1 = 0.8036$, $p_2 = 100 \times 0.8036 = 80.36$ lbf/in^2 (5.541×10^5 N/m^2).

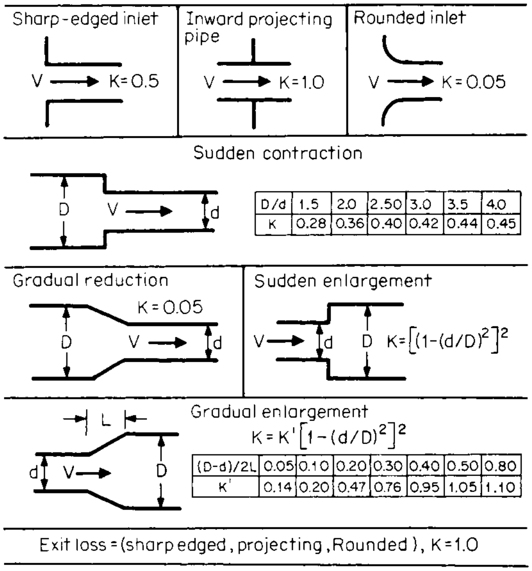
PIPING SYSTEMS

Resistance Parameters The resistance to flow of a piping system is similar to the resistance of an object immersed in a flow stream and is made up of pressure (inertia) or shape drag and skin-friction (viscous) drag. For long, straight pipes the pressure drag is characterized by the **relative roughness** ε/D and the skin friction by the **Reynolds number R.** For other piping components, two parameters are used to describe the resistance to flow, the **resistance coefficient** $K = fL/D$ and the **equivalent length** $L/D = K/f$. The resistance-coefficient method assumes that the component loss is all due to pressure drag and that the flow through the component is completely turbulent and independent of Reynold's number. The equivalent-length method assumes that resistance of the component varies in the same manner as does a straight pipe. The basic assumption then is that its pressure drag is the same as that for the relative roughness ε/D of the pipe and that the friction drag varies with the Reynolds number **R** in the same manner as the straight pipe. Both methods have the inherent advantage of simplicity in application, but neither is correct except in the fully developed turbulent region. Two excellent sources of information on the resistance of piping-system components are the Hydraulic Institute ''Pipe Friction Manual,'' which uses the resistance-coefficient method, and the Crane Company Technical Paper 410 (''Fluid Meters,'' 6th ed. ASME, 1971), which uses the equivalent-length concept.

For valves, branch flow through tees, and the type of components listed in Table 3.3.10, the pressure drag is predominant, is ''rougher'' than the pipe to which it is attached, and will extend the completely turbulent region to lower values of Reynolds number. For bends and elbows, the loss is made up of pressure drag due to the change of direction and the consequent secondary flows which are dissipated in 50 diameters or more downstream piping. For this reason, loss through adjacent bends will not be twice that of a single bend.

In long pipelines, the effect of bends, valves, and fittings is usually negligible, but in systems where there is little straight pipe, they are the controlling factor. Under-design will result in the failure of the system to deliver the required capacity. Over-design will result in inefficient operation because it will be necessary to ''throttle'' one or more of the valves. For estimating purposes, Tables 3.3.10 and 3.3.11 may be used as shown in the examples. When available, the manufacturers' data should be used, particularly for valves, because of the wide variety of designs for the same type. (See also Sec. 12.4.)

Table 3.3.10 Representative Values of Resistance Coefficient *K*



D/d	1.5	2.0	2.50	3.0	3.5	4.0
K	0.28	0.36	0.40	0.42	0.44	0.45

(D-d)/2L	0.05	0.10	0.20	0.30	0.40	0.50	0.80
K'	0.14	0.20	0.47	0.76	0.95	1.05	1.10

SOURCE: Compiled from data given in ‘‘Pipe Friction Manual,’’ 3d ed., Hydraulic Institute, 1961.

Table 3.3.11 Representative Equivalent Length in Pipe Diameters (L/D) of Various Valves and Fittings

Globe valves, fully open	450
Angle valves, fully open	200
Gate valves, fully open	13
¾ open	35
½ open	160
¼ open	900
Swing check valves, fully open	135
In line, ball check valves, fully open	150
Butterfly valves, 6 in and larger, fully open	20
90° standard elbow	30
45° standard elbow	16
90° long-radius elbow	20
90° street elbow	50
45° street elbow	26
Standard tee:	
Flow through run	20
Flow through branch	60

SOURCE: Compiled from data given in ‘‘Flow of Fluids,’’ Crane Company Technical Paper 410, ASME, 1971.

Series Systems In a single piping system made of various sizes, the practice is to group all of one size together and apply the continuity equation, as shown in the following example.

EXAMPLE. Water at 68°F (20°C) leaves an open tank whose surface elevation is 180 ft and enters a 2-in schedule 40 steel pipe via a sharp-edged entrance. After 50 ft of straight 2-in pipe that contains a 2-in globe valve, the line enlarges suddenly to an 8-in schedule 40 steel pipe which consists of 100 ft of straight 8-in pipe, two standard 90° elbows and one 8-in angle valve. The 8-in line discharges below the surface of another open tank whose surface elevation is 100 ft. Determine the volumetric flow rate.

$$D_1 = 2.067/12 = 0.1723 \text{ ft} \qquad \text{and} \qquad D_2 = 7.981/12 = 0.6651 \text{ ft}$$
$$\varepsilon/D_1 = 150 \times 10^{-6}/0.1723 = 8.706 \times 10^{-4}$$
$$\varepsilon/D_2 = 150 \times 10^{-6}/0.6651 = 2.255 \times 10^{-4}$$

For turbulent flow,

$$\frac{1}{\sqrt{f_1}} = -2 \log_{10} \left(\frac{\varepsilon/D}{3.7} \right)$$

$$\frac{1}{\sqrt{f_1}} = -2 \log_{10} \left(\frac{8.706 \times 10^{-4}}{3.7} \right) \qquad f_1 = 0.01899$$

$$\frac{1}{\sqrt{f_1}} = -2 \log_{10} \left(\frac{2.255 \times 10^{-4}}{3.7} \right) \qquad f_2 = 0.01407$$

1. *2-in components*	*K*
Entrance loss, sharp-edged	$= 0.5$
50 ft straight pipe $= f_1\,(50/0.1723)$	$= 290.2f_1$
Globe valve $= f_1\,(L/D)$	$= 450.0f_1$
Sudden enlargement $k = [-(D/D_2)^2]^2$	
$= [1 - (2.067/7.981)^2]^2$	$= 0.87$
	$\Sigma K_1 = 1.37 + 740.2f_1$

2. *8-in components*	*K*
100 ft of straight pipe $f_2\,(100/0.6651)$	$= 150.4f_2$
2 standard 90° elbows $2 \times 30\, f_2$	$= 60\ f_2$
1 angle valve $200\, f_2$	$= 200\ f_2$
Exit loss	$= 1$
	$\Sigma K_2 = 1 + 410.4f_2$

3. *Apply equation of motion*

$$h_{1f2} = z_1 - z_2 = (\Sigma K_1)\frac{V_1^2}{2g} + (\Sigma K_2)\frac{V_2^2}{2g}$$

From continuity, $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ for $\rho_1 = \rho_2$

$$V_2 = V_1(A_1/A_2) = V_1(D_1/D_2)^2$$
$$h_{1f2} = z_1 - z_2 = [\Sigma K_1 + \Sigma K_2(D_1/D_2)^4]V_1^2/2g$$
$$V_1 = \{[2g(z_1 - z_2)]/[\Sigma K_1 + \Sigma K_2(D_1/D_2)^4]\}^{1/2}$$
$$V_1 = \left[\frac{2 \times 32.17 \times (180 - 100)}{(1.37 + 740.2f_1) + (1 + 410.4f_2)(2.067/7.981)^4} \right]^{1/2}$$
$$V_1 = \frac{71.74}{(1.374 + 740.2f_1 + 1.846f_2)^{1/2}}$$

4. For first trial assume f_1 and f_2 for complete turbulence

$$V_1 = \frac{71.74}{(1.374 + 740.2 \times 0.01899 + 1.846 \times 0.01407)^{1/2}}$$
$$V_1 = 18.25 \text{ ft/s}$$
$$V_2 = 18.25\,(2.067/7.981)^2 = 1.224 \text{ ft/s}$$
$$\mathbf{R}_1 = \rho_1 V_1 D_1/\mu = (1.937)(18.25)(0.1723)/(20.92 \times 10^{-6})$$
$$\mathbf{R}_1 = 291{,}100 > 4{,}000 \therefore \text{flow is turbulent}$$
$$\mathbf{R}_2 = \rho_2 V_2 D_2/\mu_2 = (1.937)(1.224)(0.6651)/(20.92 \times 10^{-6})$$
$$\mathbf{R}_2 = 75{,}420 > 4{,}000 \therefore \text{flow is turbulent}$$

5. For second trial use first trial V_1 and V_2. From Fig. 3.3.24 and the Colebrook equation,

$$\frac{1}{\sqrt{f_1}} = -2 \log_{10} \left(\frac{8.706 \times 10^{-4}}{3.7} + \frac{2.51}{291{,}100\,\sqrt{0.020}} \right)$$
$$f_1 = 0.02008$$
$$\frac{1}{\sqrt{f_2}} = -2 \log_{10} \left(\frac{2.255 \times 10^{-4}}{3.7} + \frac{2.51}{75{,}420\,\sqrt{0.020}} \right)$$
$$f_2 = 0.02008$$
$$V_1 = \frac{71.74}{(1.374 + 740.2 \times 0.02008 + 1.864 \times 0.02008)^{1/2}}$$
$$V_1 = 17.78$$

A third trial results in $V = 17.77$ ft/s or $Q = A_1V_1 = (\pi/4)(0.1723)^2(17.77) = 0.4143$ ft³/s (1.173×10^{-2} m³/s).

Parallel Systems In solution of problems involving two or more parallel pipes, the head loss for all of the pipes is the same as shown in the following example.

EXAMPLE. Benzene at 68°F (20°C) flows at a rate of 0.5 ft³/s through two parallel straight, horizontal pipes connecting two pressurized tanks. The pipes are both schedule 40 steel, one being 1 in, the other 2 in. They both are 100 ft long and have connections that project inwardly in the supply tank. If the pressure in the supply tank is maintained at 100 lbf/in², what pressure should be maintained on the receiving tank?

$$D_1 = 1.049/12 = 0.08742 \text{ ft} \qquad \text{and} \qquad D_2 = 2.067/12 = 0.1723 \text{ ft}$$
$$\varepsilon/D_1 = 150 \times 10^{-6}/0.08742 = 1.716 \times 10^{-3}$$
$$\varepsilon/D_2 = 150 \times 10^{-6}/0.1723 = 8.706 \times 10^{-4}$$

For turbulent flow,

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D}{3.7}\right)$$

$$\frac{1}{\sqrt{f_1}} = -2\log_{10}\left(\frac{1.716\times10^{-3}}{3.7}\right) \qquad f_1 = 0.02249$$

$$\frac{1}{\sqrt{f_2}} = -2\log_{10}\left(\frac{8.706\times10^{-4}}{3.7}\right) \qquad f_2 = 0.01899$$

1. *1-in. components* — K
   - Entrance loss, inward projection $= 1.0$
   - 100 ft straight pipe f_1 (100/0.08742) $= 1{,}144\, f_1$
   - Exit loss $= 1.0$
   - $\Sigma K_1 = 2.0 + 1{,}144\, f_1$
2. *2-in components*
   - Entrance loss, inward projection $= 1.0$
   - 100 ft straight pipe f_2 (100/0.1723) $= 580.4\, f_2$
   - Exit loss $= 1.0$
   - $\Sigma K_2 = 2.0 + 580.4\, f_2$

$h_f = \Sigma K_1\, V_1^2/2g = \Sigma K_2\, V_2^2/2g$
From the continuity equation, $Q = AV$

$$\Sigma K_1 \frac{Q_1^2}{2gA_1^2} = \Sigma K_2 \frac{Q_2^2}{2gA_2^2}$$

Solving for Q_1/Q_2,

$$\frac{Q_1}{Q_2} = \frac{A_1}{A_2}\sqrt{\frac{\Sigma K_2}{\Sigma K_1}} = \left(\frac{D_1}{D_2}\right)^2\sqrt{\frac{\Sigma K_2}{\Sigma K_1}} = \left(\frac{D_1}{D_2}\right)^2\sqrt{\frac{2.0 + 580.4\, f_2}{2.0 + 1{,}144\, f_1}}$$

For first trial assume flow is completely turbulent,

$$\frac{Q_1}{Q_2} = \left(\frac{0.08742}{0.1723}\right)^2\sqrt{\frac{2.0 + 580.4\times0.01899}{2.0 + 1{,}144\times0.02249}}$$

$$\frac{Q_1}{Q_2} = 0.1764 \qquad Q = Q_1 + Q_2 = 0.1764\, Q_2 + Q_2$$

$$0.5000 = 1.1764\, Q_2 \qquad Q_2 = 0.4250$$
$$Q_1 = 0.5000 - 0.4250 = 0.0750$$

for the second trial use first-trial values,

$V_1 = Q_1/A_1 = 0.0750/(\pi/4)(0.08742)^2 = 12.50$
$V_2 = Q_2/A_2 = 0.4250/(\pi/4)(0.1723)^2 = 18.23$
$\mathbf{R}_1 = \rho_1 V_1 D_1/\mu_1 = (1.705)(12.50)(0.08742)/(13.62\times10^{-6})$
$\mathbf{R}_1 = 136{,}800 > 4{,}000 \therefore$ flow is turbulent
$\mathbf{R}_2 = \rho_2 V_2 D_2/\mu_2 = (1.705)(18.23)(0.1723)/(13.62\times10^{-6})$
$\mathbf{R}_2 = 393{,}200 > 4{,}000 \therefore$ flow is turbulent

Using the Colebrook equation and Fig. 3.3.24,

$$\frac{1}{\sqrt{f_1}} = -2\log_{10}\left(\frac{1.716\times10^{-3}}{3.7} + \frac{2.51}{136{,}800\sqrt{0.024}}\right)$$
$$f_1 = 0.02389$$
$$\frac{1}{\sqrt{f_2}} = -2\log_{10}\left(\frac{8.706\times10^{-4}}{3.7} + \frac{2.51}{393{,}200\sqrt{0.020}}\right)$$
$$f_2 = 0.01981$$
$$h_f = \Sigma K_1\frac{V_1^2}{2g} = \Sigma K_2\frac{V_2^2}{2g}$$

$\Sigma K_1 V_1^2/2g = (2.0 + 1{,}144\times0.02389)(12.50)^2/(2\times32.17) = 71.23$
$\Sigma K_2 V_2^2/2g = (2.0 + 580.4\times0.01981)(18.23)^2/(2\times32.17) = 69.80$

71.23 = 69.80; further trials not justifiable because of accuracy of f, K, L/D. Use average or 70.52, so that $\Delta p = \rho g h_f = (1.705\times32.17\times70.52)/144 = 26.86$ lbf/in² $= p_1 - p_2 = 100 - p_2$, $p_2 = 100 - 26.86 = 73.40$ lbf/in² (5.061×10^5 N/m²).

Branch Flow Problems of a single line feeding several points may be solved as shown in the following example.

EXAMPLE. Ethyl alcohol at 68°F (20°C) flows from tank *A*, which is maintained at a constant pressure of 100 lb/in² through 200 ft of 2-in cast-iron schedule 40 pipe to a Y branch connection ($K = 0.5$) where 100 ft of 2-in pipe goes to tank *B*, which is maintained at 80 lbf/in² and 50 ft of 2-in pipe to tank *C*, which is also maintained at 80 lbf/in². All tank connections are flush and sharp-edged and are at the same elevation. Estimate the flow rate to each tank.

$$D = 2.067/12 = 0.1723 \text{ ft}$$
$$\varepsilon/D = 850\times10^{-6}/0.1723 = 4.933\times10^{-3}$$

For turbulent flow, $\dfrac{1}{\sqrt{f}} = -2\log_{10}\left(\dfrac{\varepsilon/D}{3.7}\right)$

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{4.933\times10^{-3}}{3.7}\right) \qquad f = 0.03025$$

$h_{AfB} = (p_A - p_B)/\rho g = 144(100 - 80)/(1.532\times32.17) = 58.44$
$h_{AfC} = (p_A - p_C)/\rho g = h_{AfB} = 58.44$

Let point *X* be just before the Y; then

1. *From tank A to Y* — K
   - Entrance loss, sharp-edged $= 0.5$
   - 200 ft straight pipe $= f_{AX}(200/0.1723)$ $= 1{,}161\, f_{AX}$
   - $\Sigma K_{AX} = 0.5 + 1{,}161\, f_{AX}$
2. *From Y to tank* B
   - Y branch $= 0.5$
   - 100 ft straight pipe $= f_{XB}(100/0.1723)$ $= 580.4\, f_{XB}$
   - Exit loss $= 1.0$
   - $\Sigma K_{XB} = 1.5 + 580.4\, f_{XB}$
3. *From Y to tank* C
   - Y branch $= 0.5$
   - 50 ft straight pipe $= f_{XC}(50/0.1723)$ $= 290.2\, f_{XC}$
   - Exit loss $= 1.0$
   - $\Sigma K_{XC} = 1.5 + 290.2\, f_{XC}$

Balance of flows:

$$Q_{AX} = Q_{XB} + Q_{XC}$$

and from continuity, ($A_{AX} = A_{XB} = A_{XC}$), $V_{AX} = V_{XB} + V_{XC}$; then

$$h_{AfB} = \Sigma K_{AX}\frac{V_{AX}^2}{2g} + \Sigma K_{XB}\frac{V_{XB}^2}{2g}$$
$$h_{AfC} = \Sigma K_{AX}\frac{V_{AX}^2}{2g} + \Sigma K_{XC}\frac{V_{XC}^2}{2g}$$

For first trial assume completely turbulent flow

$$h_{AfB} = \frac{(0.5 + 1{,}161\, f_{AX})V_{AX}^2}{2g} + \frac{(1.5 + 580.4\, f_{XB})V_{XB}^2}{2g}$$
$$58.44 = \frac{(0.5 + 1{,}161\times0.03025)V_{AX}^2}{2\times32.17} + \frac{(1.5 + 580.4\times0.03025)V_{XB}^2}{2\times32.17}$$
$$58.44 = 0.5536\, V_{AX}^2 + 0.2962\, V_{XB}^2$$

and in a like manner

$$h_{AfC} = 58.44 = 0.5536\, V_{XC}^2 + 0.1598\, V_{XC}^2$$

Equating $h_{AfB} = h_{AfC}$,

$$0.5536\, V_{AX}^2 + 0.2962\, V_{XB}^2 = 0.5536\, V_{AX}^2 + 0.1598\, V_{XC}^2$$

or $\quad V_{XC} = 1.3615\, V_{XB}$ and since $V_{AX} = V_{XB} + V_{XC}$
$\quad V_{AX} = V_{XB} + 1.3615\, V_{XB} = 2.3615\, V_{XB}$

so that $\quad h_{AfB} = 58.44 = 0.5536(2.3615\, V_{XB}^2) + 0.2962\, V_{XB}^2$
$\quad V_{XB} = 4.156$
$\quad V_{XC} = 1.3615(4.156) = 5.658$
$\quad V_{AX} = 4.156 + 5.658 = 9.814$

Second trial,

$$\mathbf{R}_{AX} = \frac{\rho V_{AX} D}{\mu} = \frac{1.532\times9.814\times0.1723}{25.06\times10^{-6}}$$
$$\mathbf{R}_{AX} = 103{,}400 > 4{,}000 \therefore \text{flow is turbulent}$$

In a like manner,

$$\mathbf{R}_{XB} = 43{,}780 \qquad \mathbf{R}_{XC} = 59{,}600$$

Using the Colebrook equation and Fig. 3.3.24,

$$\frac{1}{\sqrt{f_{AX}}} = -2\log_{10}\left(\frac{4.933\times10^{-3}}{3.7} + \frac{2.51}{103{,}400\sqrt{0.031}}\right)$$
$$f_{AX} = 0.03116$$

In a like manner,

$$f_{XB} = 0.03231 \qquad f_{XC} = 0.03179$$

$$h_{AfB} = \frac{(0.5 + 1{,}161 \times 0.03116)V_{AX}^2}{2 \times 32.17} + \frac{(1.5 + 580.4 \times 0.03231)V_{XB}^2}{2 \times 32.17}$$

$$h_{AfB} = 0.5700\ V_{AX}^2 + 0.3148\ V_{XB}^2$$

$$h_{AfC} = 0.5700\ V_{AX}^2 + \frac{(1.5 + 290.2 \times 0.03179)V_{XC}^2}{2 \times 32.17}$$

$$h_{AfC} + 0.5700\ V_{AX}^2 + 0.1667\ V_{XC}^2$$
$$0.3148\ V_{XB}^2 = 0.1667\ V_{XC}^2$$
$$V_{XC} = 1.374\ V_{XB}$$
$$V_{AX} = V_{XB} + 1.374\ V_{XB} = 2.374\ V_{XB}$$

so that

$$h_{AfB} = 58.44 = 0.5700\ (2.374\ V_{XB})^2 + 0.3148\ V_{XB}^2$$
$$V_{XB} = 4.070 \qquad V_{XC} = 5.592 \qquad V_{AX} = 9.663$$

Further trials are not justified.

$$A = \pi D^2/4 = (\pi/4)(0.1723)^2 = 0.02332\ \text{ft}^2$$
$$Q_{AB} = V_{AB}A = 4.070 \times 0.02332 = 0.09491\ \text{ft}^1/\text{s}\ (2.686 \times 10^{-1}\ \text{m}^1/\text{s})$$
$$Q_{XC} = V_{XC}A = 5.592 \times 0.02332 = 0.1304\ \text{ft}^3/\text{s}\ (3.693 \times 10^{-3}\ \text{m}^3/\text{s})$$

Siphons are arrangements of hose or pipe which cause liquids to flow from one level A in Fig. 3.3.25 to a lower level C over an intermediate summit B. Performance of siphons may be evaluated from the equation of motion between points A and B:

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B + h_{AfB}$$

Noting that on the surface $V_A = 0$ and the minimum pressure that can exist at point B is the vapor pressure pv, the maximum elevation of point B is

$$z_B - z_A = \frac{p_A}{\gamma} - \left(\frac{p_v}{\gamma} + \frac{V_B^2}{2g} + h_{AfB}\right)$$

The friction loss $h_f = \Sigma K_{AB} V_B^2/2g$, and let $V_B = V$; then

$$z_B - z_A = \frac{p_A - p_v}{\rho g} - (1 - \Sigma K_{AB})\frac{V^2}{2g}$$

Flow under this maximum condition will be uncertain. The air pump or ejector used for priming the pipe (flow will not take place unless the siphon is full of water) might have to be operated occasionally to remove accumulated air and vapor. Values of $z_B - z_A$ less than those calculated by the above equation should be used.

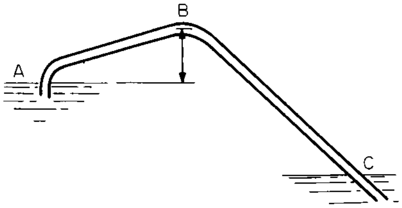


Fig. 3.3.25 Siphon.

EXAMPLE. The siphon shown in Fig. 3.3.25 is composed of 2,000 ft of 6-in schedule 40 cast-iron pipe. Reservoir A is at elevation 800 ft and C at 600 ft. Estimate the maximum height for $z_B - z_A$ if the water temperature may reach 104°F (40°C), and the amount of straight pipe from A to B is 100 ft. For the first bend $L/D = 25$ and the second (at B) $L/D = 50$. Atmospheric pressure is 14.70 lbf/in². For 6-in schedule 40 pipe $D = 6.065/12 = 0.5054$ ft, $\varepsilon/D = 850 \times 10^{-6}/0.5054 = 1.682 \times 10^{-3}$. Turbulent friction factor

$$1/\sqrt{f} = -2\log_{10}\left(\frac{\varepsilon/D}{}\right) = -2\log_{10}(1.682 \times 10^{-3}/3.7) = 0.02238$$

1. *Components from* A *to* B. (Note loss in second bend takes place in downstream piping.) K
   - Entrance (inward projection) $= 1.0$
   - 100 ft straight pipe f (100/0.5054) $= 197.9\,f$
   - First bend $= 25\ f$
   - $\Sigma K_{AB} = 1.0 + 227.9\,f$
2. *Components from* A *to* C
   - ΣK_{AB} $= 1.0 + 2{,}229\,f$
   - 1,900 ft of straight pipe f (1,900/0.5054) $= 3{,}759.4\,f$
   - Second bend $= 50\,f$
   - Exit loss $= 1$
   - $\Sigma K_{AC} = 2.0 + 4{,}032\,f$

First trial assume complete turbulence. Writing the equation of motion between A and C.

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_C}{\gamma} + \frac{V_C^2}{2g} + z_C + \Sigma K_{AC}\frac{V^2}{2g}$$

Noting $V_A = V_C = 0$, and $p_A = p_C = 14.7$ lbf/in²,

$$V = \sqrt{\frac{2g(z_A - z_C)}{\Sigma K_{AC}}} = \sqrt{\frac{2g(z_A - z_C)}{2.0 + 4{,}032\,f}}$$
$$= \sqrt{\frac{2 \times 32.17\ (800 - 600)}{2.0 + 4{,}032\,f}}$$
$$= \frac{113.44}{\sqrt{2.0 + 4{,}032 \times 0.02238}}$$
$$= 11.81$$

Second trial, use first-trial values,

$$\mathbf{R} = \frac{\rho VD}{\mu} = (1.925)(11.81)(0.5054)/(13.61 \times 10^{-6})$$
$$\mathbf{R} = 846{,}200 > 4{,}000 \therefore \text{flow is turbulent}$$

From Fig. 3.3.24 and the Colebrook equation,

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{1.682 \times 10^{-3}}{3.7} + \frac{2.51}{844{,}200\ \sqrt{0.023}}\right)$$
$$f = 0.02263$$
$$V = \frac{113.44}{\sqrt{2.0 + 4{,}032 \times 0.02263}} = 11.75 \qquad \text{(close check)}$$

From Sec. 4.2 steam tables at 104°F, $p_v = 1.070$ lbf/in², the maximum height

$$z_B - z_A = \frac{p_A - p_v}{\rho g} - (1 + \Sigma K_{AB})\frac{V^2}{2g}$$
$$= \frac{144(14.70 - 1.070)}{1.925 \times 32.17} - (1 + 1 + 227.9$$
$$\times 0.02262)\frac{(11.75)^2}{2 \times 32.17} = 16.58\ \text{ft}\ (5.053\ \text{m})$$

Note that if a ± 10 percent error exists in calculation of pressure loss, maximum height should be limited to ~ 15 ft (5 m).

ASME PIPELINE FLOWMETERS

Parameters Dimensional analysis of the flow of an incompressible fluid flowing in a pipe of diameter D, surface roughness ε, through a primary element (venturi, nozzle or orifice) whose diameter is d with a velocity of V, producing a pressure drop of Δp sensed by pressure taps located a distance L apart results in $f(\mathbf{C}_p, \mathbf{R}_d, \varepsilon/D, d/D) = 0$, which may be written as $\Delta p = \mathbf{C}_p \rho V^2/2$. Conventional practice is to express the relations as $V = \mathbf{K}\sqrt{2\Delta p/\rho}$, where $\mathbf{K}$ is the **flow coefficient,** $\mathbf{K} = 1/\sqrt{\mathbf{C}_p}$, and $\mathbf{K} = f(\mathbf{R}_d, L/D, \varepsilon/d, d/D)$. The ratio of the diameter of the primary element to meter tube (pipe) diameter D is known as the **beta ratio,** where $\beta = d/D$. Application of the continuity equation leads to $Q = \mathbf{K}A_2\sqrt{2\Delta p/\rho}$, where A_2 is the area of the primary element.

Conventional practice is to base flowmeter computations on the assumption of one-dimensional frictionless flow of an incompressible fluid in a horizontal meter tube and to correct for actual conditions by the use of a coefficient for viscous effects and a factor for elastic effects. Application of the Bernoulli equation for horizontal flow from section 1 (inlet tap) to section 2 (outlet tap) results in $p_1/\rho g + V_1^2/2g = p_2/\rho g + V_2^2/2g$ or $(p_1 - p_2)/\rho = V_2^2 - V_1^2 = \Delta p/\rho$. From the equation of continuity, $Q_i = A_1V_1 = A_2V_2$, where Q_i is the **ideal flow rate.** Substituting, $2\Delta p/\rho = Q_i^2/A_1^2 - Q_i^2/A_2^2$, and solving for Q_i, $Q_i = A_2\sqrt{2\Delta p/\rho}/\sqrt{1 - (A_2/A_1)^2}$, noting that $A_2/A_1 = (d/D)^2 = \beta^2$, $Q_i = A_2\sqrt{2\Delta p/\rho}/\sqrt{1 - \beta^4}$. The **discharge coefficient** C is defined as the ratio of the actual flow Q to the ideal flow Q_i, or $C = Q/Q_i$, so that $Q = CQ_i = CA_2\sqrt{2\Delta p/\rho}/\sqrt{1 - \beta^4}$. It is customary to write the volumetric-flow equation as $Q = CEA_2\sqrt{2\Delta p/\rho}$, where $E = 1/\sqrt{1 - \beta^4}$. E is called the **velocity-of-approach factor** because it accounts for the one-dimensional kinetic energy at the upstream tap. Comparing the equation from dimensional analysis with the modified Bernoulli equation, $Q = \mathbf{K}A_2\sqrt{2\Delta p/\rho} = CEA_2\sqrt{2\Delta p/\rho}$, or $\mathbf{K} = CE$ and $C = f(\mathbf{R}_d, L/D, \beta)$.

For compressible fluids, the incompressible equation is modified by the **expansion factor** Y, where Y is defined as the ratio of the flow of a compressible fluid to that of an incompressible fluid at the same value of Reynolds number. Calculations are then based on inlet-tap-fluid properties, and the compressible equation becomes

$$Q_1 = \mathbf{K}YA_2\sqrt{2\Delta p/\rho_1} = CEYA_2\sqrt{2\Delta p/\rho_1}$$

where $Y = f(L/D, \varepsilon/D, \beta, \mathbf{M})$. Reynolds number $\mathbf{R}_d$ is also based on inlet-fluid properties, but on the primary-element diameter or

$$\mathbf{R}_d = \rho_1V_2d/\mu_1 = \rho_1(Q_1/A_2)d/\mu_1 = 4\rho_1Q_1/\pi\, d\mu_1$$

Caution The numerical values of coefficients for flowmeters given in the paragraphs to follow are based on experimental data obtained with long, straight pipes where the velocity profile approaching the primary element was fully developed. The presence of valves, bends, and fittings upstream of the primary element can cause serious errors. For approach and discharge, straight-pipe requirements, ‘‘Fluid Meters,’’ (6th ed., ASME, 1971) should be consulted.

Venturi Tubes Figure 3.3.26 shows a typical venturi tube consisting of a cylindrical inlet, convergent cone, throat, and divergent cone. The convergent entrance has an included angle of about 21° and the divergent cone 7 to 8°. The purpose of the divergent cone is to reduce the overall pressure loss of the meter; its removal will have no effect on the coefficient of discharge. Pressure is sensed through a series of holes in the inlet and throat. These holes lead to an annular chamber, and the two chambers are connected to a pressure-differential sensor. Discharge coefficients for venturi tubes as established by the American Society of Mechanical Engineers are given in Table 3.3.12. Coefficients of discharge outside the tabulated limits must be determined by individual calibrations.

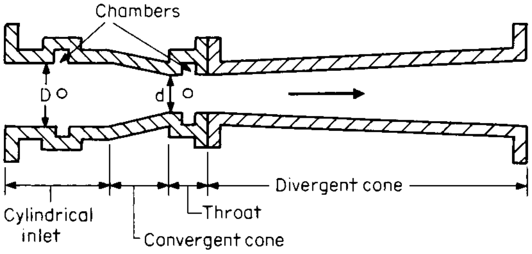


Fig. 3.3.26 Venturi tube.

EXAMPLE. Benzene at 68°F (20°C) flows through a machined-inlet venturi tube whose inlet diameter is 8 in and whose throat diameter is 3.5 in. The differential pressure is sensed by a U-tube manometer. The manometer contains mercury under the benzene, and the level of the mercury in the throat leg is 4 in. Compute the volumetric flow rate. Noting that $D = 8$ in (0.6667 ft) and $\beta = 3.5/8 = 0.4375$ are within the limits of Table 3.3.12, assume $C = 0.995$, and then check $\mathbf{R}_d$ to verify if it is within limits. For a U-tube manometer (Fig. 3.3.6*a*), $p_2 - p_1 = (\gamma_m - \gamma_f)h = \Delta p$ and $\Delta p/\rho_1 = (\rho_m g - \rho_f g)h/\rho_f = g(\rho_m/\rho_f - 1)h = 32.17(26.283/1.705 - 1)(4/12) = 154.6$. For a liquid, $Y = 1$ (incompressible fluid), $E = 1/\sqrt{1 - \beta^4} = 1/\sqrt{1 - (0.4375)^4} = 1.019$.

$$\begin{aligned} Q_1 &= CEY\,A_d\,\sqrt{2\Delta p/\rho_1} \\ &= (0.995)(1.019)(\pi/4)(3.5/12)^2\,\sqrt{2 \times 154.6} \\ &= 1.192\ \text{ft}^3/\text{s}\ (3.373 \times 10^{-3}\ \text{m}^3/\text{s}) \end{aligned}$$

$\mathbf{R}_d = 4\rho_1Q_1/\pi\, d\mu_1 = 4(1.705)(1.192)/\pi(3.5/12)(13.62 \times 10^{-6})$
$\mathbf{R}_d = 651{,}400$, which lies between 200,000 and 1,000,000 of Table 3.3.12 $\therefore$ solution is valid.

Flow Nozzles Figure 3.3.27 shows an ASME flow nozzle. This nozzle is built to rigid specifications, and pressure differential may be sensed by either throat taps or pipe-wall taps. Taps are located one pipe diameter upstream and one-half diameter downstream from the nozzle inlet. Discharge coefficients for ASME flow nozzles may be computed from $C = 0.9975 - 0.00653\,(10^6/\mathbf{R}_d)^a$, where $a = 1/2$ for $\mathbf{R}_d < 10^6$ and $a = 1/5$ for $\mathbf{R}_d > 10^6$. Most of the data were obtained for D between 2 and 15.75 in, $\mathbf{R}_d$ between 10^4 and 10^6, and beta between 0.15 and 0.75. For values of C within these ranges, a tolerance of 2 percent may be anticipated, and outside these limits, the tolerance may be greater than 2 percent. Because slight variations in form or dimension of either pipe or nozzle may affect the observed pressures, and thus cause the exponent a and the slope term ($-$ 0.00653) to vary considerably, nozzles should be individually calibrated.

EXAMPLE. An ASME flow nozzle is to be designed to measure the flow of 400 gal/min of 68°F (20°C) water in a 6-in schedule 40 (inside diameter = 6.065 in) steel pipe. The pressure differential across the nozzle is not to exceed 75 in of water. What should be the throat diameter of the nozzle? $\Delta p = h\rho_1 g$, $\Delta p/\rho_1 = hg = (75/12)(32.17) = 201.1$, $Q = (400/60)(231/1{,}728) = 0.8912$ ft³/s. A trial-and-error solution is necessary to establish the values of C and E because they are dependent upon β and $\mathbf{R}_d$, both of which require that d be known. Since $\mathbf{K} = CE \approx 1$, assume for first trial that $CE = 1$. Since a liquid is involved, $Y = 1$, $A_2 = Q_1/(CE)(Y)\,\sqrt{2\Delta p/\rho_1} = (0.8912)/(1)(1)\,\sqrt{2 \times 201.1} = 0.04444$ ft², $d = \sqrt{4A_2/\pi} = \sqrt{4(0.04444)/\pi} = 0.2379$ ft or $d = 0.2379 \times 12 = 2.854$ in, $\beta = d/D = 2.854/6.065 = 0.4706$.

For second trial use first-trial value:

$$E = 1/\sqrt{1 - \beta^4} = 1/\sqrt{1 - (0.4706)^4} = 1.025$$

$\mathbf{R}_d = 4\rho_1Q_1/\pi d\mu_1 = 4(1.937)(0.8912)/\pi(0.2379)(20.92 \times 10^{-6}) = 442{,}600 < 10^6 \therefore a = 1/2$ and $C = 0.9975 - 0.00653(10^6/\mathbf{R}_d)^{1/2}$. $C = 0.9975 - 0.00653(10^6/442{,}600)^{1/2} = 0.9877$. $A_2 = (0.8912)/(0.9877 \times 1.025)\sqrt{2 \times 201.1} = 0.04389$, $d_2 = \sqrt{4 \times (0.04389/\pi)} = 0.2364$, $d_2 = 0.2364 \times 12 = 2.837$ in (7.205×10^{-2} m). Further trials are not necessary in view of the ± 2 percent tolerance of C.

Table 3.3.12 ASME Coefficients for Venturi Tubes

Type of inlet cone	Reynolds number $\mathbf{R}_d$		Inlet diam D in (2.54×10^{-2} m)		β			Tolerance, %
	Min	Max	Min	Max	Min	Max	C	
Machined	5×10^5	1×10^6	2	10	0.4	0.75	0.995	±1.0
Rough welded sheet metal		2×10^6	8	48		0.70	0.985	±1.5
Rough cast			4	32	0.3	0.75	0.984	±0.7

SOURCE: Compiled from data given in ‘‘Fluid Meters,’’ 6th ed., ASME, 1971.

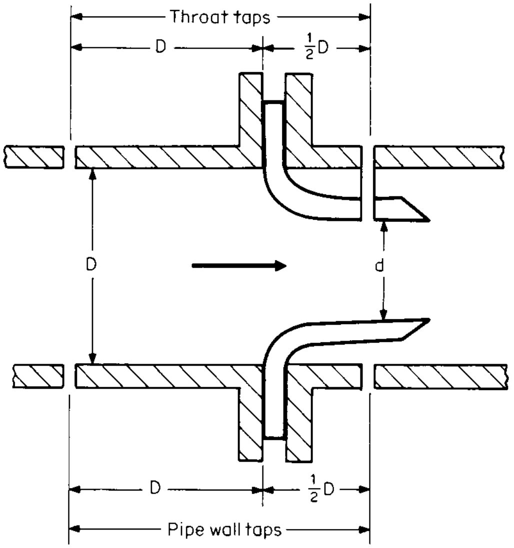


Fig. 3.3.27 ASME flow nozzle.

Compressible Flow—Venturi Tubes and Flow Nozzles The expansion factor Y is computed based on the assumption of a frictionless adiabatic (isentropic) expansion of an ideal gas from the inlet to the throat of the primary element, resulting in (see Sec. 4.1)

$$Y = \left[\frac{kr^{2/k}(1 - r^{(k-1)/k})(1 - \beta^4)}{(1 - r)(k - 1)(1 - \beta^4 r^{2/k})} \right]^{1/2}$$

where $r = p_2/p_1$.

Maximum flow is obtained when the critical pressure ratio is reached. The critical pressure ratio r_c may be calculated from

$$r^{(1-k)/k} + \frac{k-1}{2}\beta^4 r^{2/k} = \frac{k+1}{2}$$

Table 3.3.13 gives selected values of Y_c and r_c.

EXAMPLE. A piping system consists of a compressor, a horizontal straight length of 2-in-inside-diameter pipe, and a 1-in-throat-diameter ASME flow nozzle attached to the end of the pipe, discharging into the atmosphere. The compressor is operated to maintain a flow of air with 115 lbf/in^2 and 140°F (60°C) conditions in the pipe just one pipe diameter before the nozzle inlet. Barometric pressure is 14.7 lbf/in^2. Estimate the flow rate of the air in lbm/s.

From the equation of state, $\rho_1 = p_1/g_c RT_1 = (144 \times 115)/(32.17)(53.34)(140 + 459.7) = 0.01609$ slug/ft^3, $\beta = d/D = 1/2 = 0.5$, $E = 1/\sqrt{1 - \beta^4} = 1/\sqrt{1 - (0.5)^4} = 1.033$, $r = p_2/p_1 = 14.7/115 = 0.1278$, but from Table 3.3.13 at $\beta = 0.5$, $k = 1.4$, $r_c = 0.5362$, and $Y_c = 0.6973$, so that because of critical flow the throat pressure $p_c = 115 \times 0.5362 = 61.66$ lbf/in^2. $\Delta p_c/\rho_1 = 144(115 - 61.66)/0.01609 = 477{,}375$. A trial-and-error solution is necessary to obtain C. For the first trial assume $10^6/\mathbf{R}_d = 0$ or $C = 0.9975$. Then $Q_1 = CEY_c A_2 \sqrt{2\Delta p_c/\rho_1} = (0.9975)(1.033)(0.6973)(\pi/4)(1.12)^2 \sqrt{2 \times 477{,}375} = 3.829$ ft^3/s, $\mathbf{R}_d = 4\rho_1 Q/\pi\, d\mu_1 = (4)(0.01609)(3.828)/\pi(1/12)(41.79 \times 10^{-8}) = 2{,}252{,}000$.

Second trial, use first-trial values:

$$\mathbf{R} > 10^6,\ a = 115,\ C = 0.9975 - (0.00653)(10^6/2{,}252{,}000)^{1/5}$$
$$C = 0.9919,\ Q_1 = 3.828(0.9919/0.9975) = 3.806 \text{ ft}^3/\text{s}$$

Further trials are not necessary in view of ± 2 percent tolerance on C.

$$\dot{m} = Q_1 \rho_1 g = 3.806 \times 0.01609 \times 32.17 = 1.970 \text{ lbm/s } (0.8935 \text{ kg/s})$$

Orifice Meters When a fluid flows through a square-edged thin-plate orifice, the minimum-flow area is found to occur downstream from the orifice plate. This minimum area is called the vena contracta, and its location is a function of beta ratio. Figure 3.3.28 shows the relative pressure difference due to the presence of the orifice plate. Because the location of the pressure taps is vital, it is necessary to specify the exact position of the downstream pressure tap. The jet con-

Table 3.3.13 Expansion Factors and Critical Pressure Ratios for Venturi Tubes and Flow Nozzles

		Critical values		Expansion factor Y			
β	k	r_c	Y_c	$r = 0.60$	$r = 0.70$	$r = 0.80$	$r = 0.90$
0	1.10	0.5846	0.6894	0.7021	0.7820	0.8579	0.9304
	1.20	0.5644	0.6948	0.7228	0.7981	0.8689	0.9360
	1.30	0.5457	0.7000	0.7409	0.8119	0.8783	0.9408
	1.40	0.5282	0.7049	0.7568	0.8240	0.8864	0.9449
0.20	1.10	0.5848	0.6892	0.7017	0.7817	0.8577	0.9303
	1.20	0.5546	0.6946	0.7225	0.7978	0.8687	0.9359
	1.30	0.5459	0.6998	0.7406	0.8117	0.8781	0.9407
	1.40	0.5284	0.7047	0.7576	0.8237	0.8862	0.9448
0.50	1.10	0.5921	0.6817	0.6883	0.7699	0.8485	0.9250
	1.20	0.5721	0.6872	0.7094	0.7864	0.8600	0.9310
	1.30	0.5535	0.6923	0.7248	0.8007	0.8699	0.9361
	1.40	0.5362	0.6973	0.7440	0.8133	0.8785	0.9405
0.60	1.10	0.6006	0.6729		0.7556	0.8374	0.9186
	1.20	0.5808	0.6784	0.6939	0.7727	0.8495	0.9250
	1.30	0.5625	0.6836	0.7126	0.7875	0.8599	0.9305
	1.40	0.5454	0.6885	0.7292	0.8006	0.8689	0.9352
0.70	1.10	0.6160	0.6570		0.7290	0.8160	0.9058
	1.20	0.5967	0.6624	0.6651	0.7469	0.8292	0.9131
	1.30	0.5788	0.6676	0.6844	0.7626	0.8405	0.9193
	1.40	0.5621	0.6726	0.7015	0.7765	0.8505	0.9247
0.80	1.10	0.6441	0.6277		0.6778	0.7731	0.8788
	1.20	0.6238	0.6331		0.6970	0.7881	0.8877
	1.30	0.6087	0.6383		0.7140	0.8012	0.8954
	1.40	0.5926	0.6433	0.6491	0.7292	0.8182	0.9021

SOURCE: Murdock, ''Fluid Mechanics and Its Applications,'' Houghton Mifflin, 1976.

traction amounts to about 60 percent of the orifice area; so orifice coefficients are in the order of 0.6 compared with the nearly unity obtained with venturi tubes and flow nozzles.

Three pressure-differential-measuring tap locations are specified by the ASME. These are the flange, vena contracta, and the 1 D and 1/2 D. In the flange tap, the location is always 1 in from either face of the orifice plate regardless of the size of the pipe. In the vena contracta tap, the upstream tap is located one pipe diameter from the inlet face of the orifice plate and the downstream tap at the location of the vena contracta. In the 1 D and 1/2 D tap, the upstream tap is located one pipe diameter from the inlet face of the orifice plate and downstream one-half pipe diameter from the inlet face of the orifice plate.

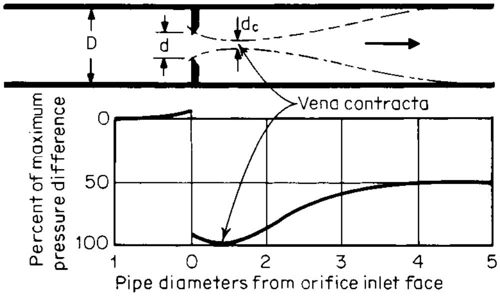


Fig. 3.3.28 Relative-pressure changes due to flow through an orifice.

Flange taps are used because they can be prefabricated, and flanges with holes drilled at the correct locations may be purchased as off-the-shelf items, thus saving the cost of field fabrication. The disadvantage of flange taps is that they are not symmetrical with respect to pipe size. Because of this, coefficients of discharge for flange taps vary greatly with pipe size.

Vena contracta taps are used because they give the maximum differential for any given flow. The disadvantage of the vena contracta tap is that if the orifice size is changed, a new downstream tap must be drilled. The 1 D and 1/2 D taps incorporate the best features of the vena contracta taps and are symmetrical with respect to pipe size.

Discharge coefficients for orifices may be calculated from

$$C = C_o + \Delta C \mathbf{R}_d^{-0.75} \qquad (\mathbf{R}_d > 10^4)$$

where C_o and ΔC are obtained from Table 3.3.14.

Tolerances for uncalibrated orifice meters are in the order of ± 1 to ± 2 percent depending upon β, D, and $\mathbf{R}_d$.

Compressible Flow through ASME Orifices As shown in Fig. 3.3.28, the minimum flow area for an orifice is at the vena contracta located downstream of the orifice. The stream of compressible fluid is not restrained as it leaves the orifice throat and is free to expand transversely and longitudinally to the point of minimum-flow area. Thus the contraction of the jet will be less for a compressible fluid than for a liquid. Because of this, the theoretical-expansion-factor equation may not be used with orifices. Neither may the critical-pressure-ratio equation be used, as the phenomenon of critical flow has not been observed during testing of orifice meters.

For orifice meters, the following equation, which is based on experimental data, is used:

$$Y = 1 - (0.41 + 0.35\beta^4)(\Delta p/p_1)/k$$

EXAMPLE. Air at 68°F (20°C) and 150 lbf/in^2 flows in a 2-in schedule 40 pipe (inside diameter = 2.067 in) at a volumetric rate of 15 ft^3/min. A 0.5500-in ASME orifice equipped with flange taps is used to meter this flow. What deflection in inches could be expected on a U-tube manometer filled with 60°F water? From the equation of state, $\rho_1 = p_1/g_cRT_1 = (144 \times 150)/(32.17)(53.34)(68 + 459.7) = 0.02385$ slug/ft^3, $\beta = 0.5500/2.067 = 0.2661$. $Q_1 = 15/60 = 0.25$ ft^3/s, $A_2 = (\pi/4)(0.5500/12)^2 = 1.650 \times 10^{-3}$ ft^2. $E = 1/\sqrt{1 - \beta^4} = 1/\sqrt{1 - (0.2661)^4} = 1.003$. $\mathbf{R}_d = 4\rho_1 Q_1/\pi\, d\mu_1 = 4(0.02385)(0.25)/\pi(0.5500/12)(39.16 \times 10^{-8})$. $\mathbf{R}_d = 423{,}000$.

From Table 3.3.14 at $\beta = 0.2661$, $D = 2.067$-in flange taps, by interpolation, $C_o = 0.5977$, $\Delta C = 9.087$, from orifice-coefficient equation $C = C_o + \Delta C\mathbf{R}_d^{-0.75}$. $C = 0.5977 + (9.087)(423{,}000)^{-0.75} = 0.5982$. A trial-and-error solution is required because the pressure loss is needed in order to compute Y. For the first trial, assume $Y = 1$, $\Delta p = (Q_1/CEYA_2)^2(\rho_1/2) = [(0.25)/(0.5982)(1.003)(Y)(1.650 \times 10^{-3})]^2(0.02385/2) = 760.5/Y^2 = 760.5/(1)^2 = 760.5$ lbf/ft^2.

Table 3.3.14 Values of C_o and ΔC for Use in Orifice Coefficient Equation

	β										
Pipe ID, in	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70
	ΔC, all taps										
All	5.486	8.106	11.153	14.606	18.451	22.675	27.266	32.215	37.513	45.153	49.129
	C_o, vena contracta and 1D and ½D taps										
All	0.5969	0.5975	0.5983	0.5992	0.6003	0.6016	0.6031	0.6045	0.6059	0.6068	0.6069
	C_o, flange taps										
2.0	0.5969	0.5975	0.5982	0.5992	0.6003	0.6016	0.6030	0.6044	0.6056	0.6065	0.6066
2.5	0.5969	0.5975	0.5983	0.5993	0.6004	0.6017	0.6032	0.6046	0.6059	0.6068	0.6068
3.0	0.5969	0.5975	0.5983	0.5993	0.6004	0.6017	0.6031	0.6044	0.6055	0.6061	0.6057
3.5	0.5969	0.5975	0.5983	0.5993	0.6004	0.6016	0.6030	0.6042	0.6052	0.6056	0.6049
4.0	0.5969	0.5976	0.5983	0.5993	0.6004	0.6016	0.6029	0.6041	0.6050	0.6052	0.6043
5.0	0.5969	0.5976	0.5983	0.5993	0.6004	0.6016	0.6028	0.6039	0.6047	0.6047	0.6034
6.0	0.5969	0.5976	0.5983	0.5993	0.6004	0.6016	0.6028	0.6038	0.6045	0.6044	0.6029
8.0	0.5969	0.5976	0.5984	0.5993	0.6004	0.6015	0.6027	0.6037	0.6042	0.6040	0.6022
10.0	0.5969	0.5976	0.5984	0.5993	0.6004	0.6015	0.6026	0.6036	0.6041	0.6037	0.6017
12.0	0.5970	0.5976	0.5984	0.5993	0.6004	0.6015	0.6026	0.6035	0.6040	0.6035	0.6015
16.0	0.5970	0.5976	0.5984	0.5993	0.6003	0.6015	0.6026	0.6035	0.6039	0.6033	0.6011
24.0	0.5970	0.5976	0.5984	0.5993	0.6003	0.6015	0.6025	0.6034	0.6037	0.6031	0.6007
48.0	0.5970	0.5976	0.5984	0.5993	0.6003	0.6014	0.6025	0.6033	0.6036	0.6029	0.6004
∞	0.5970	0.5976	0.5984	0.5993	0.6003	0.6014	0.6025	0.6032	0.6035	0.6027	0.6000

SOURCE: Compiled from data given in ASME Standard MFC-3M-1984 ''Measurement of Fluid Flow in Pipes Using Orifice, Nozzle and Venturi.''

For the second trial we use first-trial values.

$$Y = 1 - (0.41 + 0.35\beta^4)\,\frac{\Delta p/p_1}{k}$$

$$= 1 - [0.41 + 0.35(0.2661)^4]\,\frac{760.5/144 \times 150}{1.4} = 0.9896$$

$$\Delta p = \frac{760.5}{Y^2} = \frac{760.5}{(0.9896)^2} = 776.1\ \text{lbf/ft}^2$$

For the third trial we use second-trial values.

$$Y = 1 - [0.41 + 0.35(0.2661)^4]\,\frac{776.1/144 \times 150}{1.4} = 0.9894$$

$$\Delta p = \frac{776.1}{(0.9894)^2} = 793.3\ \text{lbf/ft}^2$$

Resubstitution does not produce any further change in Y. From the U-tube-manometer equation:

$$h = \frac{\Delta p}{\gamma_m = \gamma_f} = \frac{\Delta p}{g_c(\rho_m - \rho_f)}$$

$$= \frac{793.3}{32.17}(1.937 - 0.02385) = 12.89\ \text{ft}$$

$$= 12.89 \times 12 = 154.7\ \text{in}\ (3.929\ \text{m})$$

PITOT TUBES

Definition A Pitot tube is a device that is shaped in such a manner that it senses stagnation pressure. The name ''Pitot tube'' has been applied to two general classifications of instruments, the first being a tube that measures the impact or stagnation pressures only, and the second a combined tube that measures both impact and static pressures with a single primary instrument. The combined sensor is called a Pitot-static tube.

Tube Coefficient From Fig. 3.3.29, it is evident that the Pitot tube can sense only the stagnation pressure resulting from the local stream-tube velocity U. The local ideal velocity U_i for an incompressible fluid is obtained by the application of the Bernoulli equation ($z_S = z$), $U_i^2/2g + p/\rho g = U_S^2/2g + p_S/\rho g$. Solving for U_i and noting that by definition $U_S = 0$, $U_i = \sqrt{2(p_S - p)/\rho}$. Conventional practice is to define the **tube coefficient** C_T as the ratio of the actual stream-tube velocity to the ideal stream-tube velocity, or $C_T = U/U_i$ and $U = C_T U_i = C_T\sqrt{2\Delta p/\rho}$. The numerical value of C_T depends primarily upon its geometry. The value of C_T may be established (1) by calibration with a uniform velocity, (2) from published data for similar geometry, or (3) in the absence of other information, may be assumed to be unity.

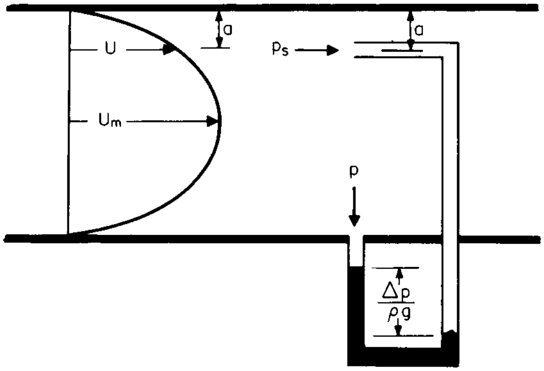


Fig. 3.3.29 Notation for Pitot tube study.

Pipe Coefficient For the calculation of volumetric flow rate, it is necessary to integrate the continuity equation, $Q = \int U\,da = AV$. The **pipe coefficient** C_P is defined as the ratio of the average velocity to the stream-tube velocity, or $C_P = V/U$, and $Q = C_P A_1 V = C_P C_T A_1\sqrt{2\Delta p/\rho}$. The numerical value of C_P is dependent upon the location of the tube and the **velocity profile.** The values of C_P may be established by (1) making a ''traverse'' by taking data at various points in the flow stream and determining the velocity profile experimentally (see ''Fluid Meters,'' 6th ed., ASME, 1971, for locations of traverse points), (2) using standard velocity profiles, (3) locating the Pitot tube at a point where $U = V$, and (4) assuming one-dimensional flow of $C_P = 1$ **only** in the absence of other data.

Compressible Flow For compressible flow, the **compression factor** Z is based on the assumption of a frictionless adiabatic (isentropic) compression of an ideal gas from the moving stream tube to the stagnation point (see Sec. 4.1), which results in

$$Z = \left[\frac{k}{k-1}\,\frac{(p_S/p)^{(k-1)/k} - 1}{(p_S/p) - 1}\right]^{1/2}$$

and the volumetric flow rate becomes

$$Q = C_P C_T Z A_1\sqrt{2\Delta p/\rho}$$

EXAMPLE. Carbon dioxide flows at 68°F (20°C) and 20 lbf/in^2 in an 8-in schedule 40 galvanized-iron pipe. A Pitot tube located on the pipe centerline indicates a pressure differential of 6.986 lbf/in^2. Estimate the mass flow rate. For 8-in schedule 40 pipe $D = 7.981/12 = 0.6651$, $\varepsilon/D = 500 \times 10^{-6}/0.6651 = 7.518 \times 10^{-4}$, $A_1 = \pi D^2/4 = (\pi/4)(0.6651)^2 = 0.3474\ \text{ft}^2$, $p_S = p + \Delta p = 20 + 6.986 = 26.986\ \text{lbf/in}^2$. From the equation of state, $\rho = p/g_c RT_o = (20 \times 144)/(32.17)(35.11)(68 + 459.7) = 0.004832$,

$$Z = \left[\frac{k}{k-1}\,\frac{(p_S/p)^{(k-1)/k} - 1}{(p_S/p) - 1}\right]^{1/2}$$

$$= \left\{[1.3/(1.3 - 1)] \times \frac{(26.986/20)^{(1.3-1)/1.3} - 1}{(26.986/20) - 1}\right\}^{1/2} = 0.9423$$

In the absence of other data, C_T may be assumed to be unity. A trial-and-error solution is necessary to determine C_P, since f requires flow rate. For the first trial assume complete turbulence.

$$1/\sqrt{f} = -2\log_{10}(7.518 \times 10^{-4}/3.7) \qquad \sqrt{f} = 0.1354$$

$$C_P = V/U = V/U_{\max} = 1/(1 + 1.43\sqrt{f}) = 1/(1 + 1.43 \times 0.1354) = 0.8378$$

$$V = C_P C_T Z\sqrt{2\Delta p/\rho} = (0.8378)(1)(0.9423)\sqrt{2 \times 144(6.987)/(0.004832)}$$

$$V = 509.4\ \text{ft/s}$$

$$\mathbf{R} = \rho VD/\mu = (0.004832)(509.4)(0.6651)/(30.91 \times 10^{-18})$$

$$\mathbf{R} = 5{,}296{,}000 > 4{,}000\ \therefore\ \text{flow is turbulent}$$

From the Colebrook equation and Fig. 3.3.24,

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{7.518 \times 10^{-4}}{3.7} + \frac{2.51}{5{,}296{,}000\sqrt{0.018}}\right)$$

$$\sqrt{f} = 0.1357$$

$$C_P = 1/(1 + 1.43 \times 0.1357) = 0.8375 \qquad \text{(close check)}$$

$$V = 509.4(0.8375/8378) = 509.2\ \text{ft/s}$$

From the continuity equation, $m = \rho A_1 V g_c = (0.004832)(0.3474)(509.2)(32.17) = 27.50$ lbm/s (12.47 kg/s).

ASME WEIRS

Definitions A weir is a dam over which liquids are forced to flow. Weirs are used to measure the flow of liquids in open channels or in conduits which do not flow full; i.e., there is a free liquid surface. Weirs are almost exclusively used for measuring water flow, although small ones have been used for metering other liquids. Weirs are classified according to their notch or opening as follows: (1) **rectangular notch** (original form); (2) V or **triangular notch;** (3) **trapezoidal notch,** which when designed with end slopes one horizontal to four vertical is called the **Cipolletti weir;** (4) the **hyperbolic weir** designed to give a constant coefficient of discharge; and (5) the **parabolic weir** designed to give a linear relationship of head to flow. As shown in Fig. 3.3.30, the top of the weir is the **crest** and the distance from the liquid surface to the crest h is called the **head.**

The sheet of liquid flowing over the weir crest is called the **nappe.** When the nappe falls downstream of the weir plate, it is said to be free, or **aerated.** When the width of the approach channel L_c is greater than the crest length L_w, the nappe will contract so that it will have a minimum width less than the crest length. For this reason, the weir is known as a **contracted weir.** For the special case where $L_w = L_c$, the contractions do not take place, and such weirs are known as **suppressed weirs.**

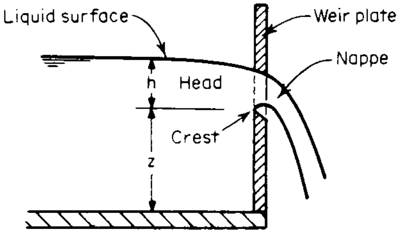


Fig. 3.3.30 Notation for weir study.

Parameters The forces acting on a liquid flowing over a weir are inertia, viscous, surface tension, and gravity. If the weir head produced by the flow is h, the characteristic length of the weir is L_w, and the channel width is L_c, either similarity or dimensional analysis leads to $f(\mathbf{F}, \mathbf{W}, \mathbf{R}, L_w/L_c) = 0$, which may be written as $V = K\sqrt{2gh}$, where K is the **weir coefficient** and $K = f(\mathbf{W}, \mathbf{R}, L_w/L_c)$. Since the weir has been almost exclusively used for metering water flow over limited temperature ranges, the effects of surface tension and viscosity have not been adequately established by experiment.

Caution The numerical values of coefficients for weirs are based on experimental data obtained from calibration of weirs with long approaches of straight channels. Head measurement should be made at a distance at least three or four times the expected maximum head h. Screens and baffles should be used as necessary to ensure steady uniform flow without waves or local eddy currents. The approach channel should be relatively wide and deep.

Rectangular Weirs Figure 3.3.31 shows a rectangular weir whose crest width is L_w. The volumetric flow rate may be computed from the continuity equation: $Q = AV = (L_w h)(K\sqrt{2gh}) = KL_w\sqrt{2g}\,h^{3/2}$. The ASME "Fluid Meters" report recommends the following equation for rectangular weirs: $Q = (2/3)CL_a\sqrt{2g}\,h_a^{3/2}$, where C is the **coefficient of discharge** $C = f(L_w/L_c, h/Z)$, L_a is the adjusted crest length $L_a = L_w + \Delta L$, and h_a is the adjusted weir head $h_a = h + 0.003$ ft. Values of C and ΔL may be obtained from Table 3.3.15. To avoid the possibility that the liquid drag along the sides of the channel will affect side contractions, $L_c - L_w$ should be at least $4h$. The minimum crest length should be 0.5 ft to prevent mutual interference of the end contractions. The minimum head for free flow of the nappe should be 0.1 ft.

EXAMPLE. Water flows in a channel whose width is 40 ft. At the end of the channel is a rectangular weir whose crest width is 10 ft and whose crest height is 4 ft. The water flows over the weir at a height of 3 ft above the crest of the weir. Estimate the volumetric flow rate. $L_w/L_c = 10/40 = 0.25$, $h/Z = 3/4 = 0.75$, from Table 3.3.15 (interpolated), $C = 0.589$, $\Delta L = 0.008$, $L_a = L_w + \Delta L = 10 + 0.008 = 10.008$ ft, $h_a = h + 0.003 = 3 + 0.003 = 3.003$ ft, $Q = (2/3)\,CL_a\sqrt{2g}\,h^{3/2}$, $Q = (2/3)(0.589)(10.008)(2 \times 32.17)^{1/2}(3.003)^{3/2}$ $Q = 164.0$ ft³/s (4.644 m³/s).

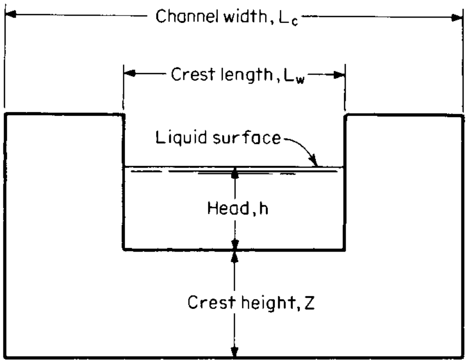


Fig. 3.3.31 Rectangular weir.

Triangular Weirs Figure 3.3.32 shows a triangular weir whose notch angle is θ. The volumetric flow rate may be computed from the continuity equation $Q = AV = (h^2 \tan\theta/2)(K\sqrt{2gh}) = K\tan(\theta/2)\sqrt{2g}\,h^{5/2}$. The ASME "Fluid Meters" report recommends the following for triangular weirs: $Q = (8/15)\,C\tan(\theta/2)\sqrt{2g}\,(h + \Delta h)^{5/2}$, where C is the coefficient of discharge $C = f(\theta)$ and Δh is the correction for head/crest ratio $\Delta h = f(\theta)$. Values of C and Δh may be obtained from Table 3.3.16.

EXAMPLE. It is desired to maintain a flow of 167 ft³/s in an open channel whose width is 20 ft at a height of 7 ft by locating a triangular weir at the end of the channel. The weir has a crest height of 2 ft. What notch angle is required to maintain these conditions? A trial-and-error solution is required. For the first trial assume $\theta = 60°$ (mean value 20 to 100°); then $C = 0.576$ and $\Delta h = 0.004$.

$$h + Z = 7 = h + 2 \therefore h = 5$$
$$Q = (8/15)\,C\tan(\theta/2)\sqrt{2g}\,(h + \Delta h)^{5/2}$$

$167 = (8/15)(0.576)\tan(\theta/2)\sqrt{2 \times 32.17}\,(5 + 0.004)^{5/2}$, $\tan^{-1}(\theta/2) = 1.20993$, $\theta = 100°51'$.

Second trial, using $\theta = 100$, $C = 0.581$, $\Delta h = 0.003$, $167 = (8/15)(0.581)\tan(\theta/2)\sqrt{2 \times 32.17}\,(5 + 0.003)^{5/2}$, $\tan^{-1}(\theta/2) = 1.20012$, $\theta = 100°39'$ (close check).

Table 3.3.15 Values of C and ΔL for Use in Rectangular-Weir Equation

	Crest length/channel width = L_w/L_c							
h/Z	0	0.2	0.4	0.6	0.7	0.8	0.9	1.0
	Coefficient of discharge C							
0	0.587	0.589	0.591	0.593	0.595	0.597	0.599	0.603
0.5	0.586	0.588	0.594	0.602	0.610	0.620	0.631	0.640
1.0	0.586	0.587	0.597	0.611	0.625	0.642	0.663	0.676
1.5	0.584	0.586	0.600	0.620	0.640	0.664	0.695	0.715
2.0	0.583	0.586	0.603	0.629	0.655	0.687	0.726	0.753
2.5	0.582	0.585	0.608	0.637	0.671	0.710	0.760	0.790
3.0	0.580	0.584	0.610	0.647	0.687	0.733	0.793	0.827
	Adjustment for crest length ΔL, ft							
Any	0.007	0.008	0.009	0.012	0.013	0.014	0.013	−0.005

SOURCE: Compiled from data given in "Fluid Meters," ASME, 1971.

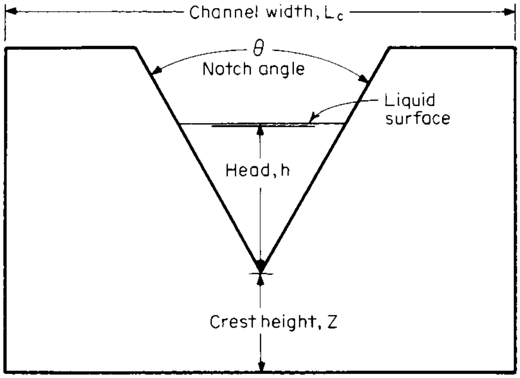


Fig. 3.3.32 Triangular weir.

Table 3.3.16 Values of C and Δh for Use in Triangular-Weir Equation

	Weir notch angle θ, deg						
Item	20	30	45	60	75	90	100
C	0.592	0.586	0.580	0.576	0.576	0.579	0.581
Δh, ft	0.010	0.007	0.005	0.004	0.003	0.003	0.003

SOURCE: Compiled from data given in "Fluid Meters," ASME, 1971.

OPEN-CHANNEL FLOW

Definitions An **open channel** is a conduit in which a liquid flows with a free surface subjected to a constant pressure. Flows of water in natural streams, artificial canals, irrigation ditches, sewers, and flumes are examples where the water surface is subjected to atmospheric pressure. The flow of any liquid in a pipe where there is a free liquid surface is an example of open-channel flow where the liquid surface will be subjected to the pressure existing in the pipe. The **slope** S of a channel is the change in elevation per unit of horizontal distance. For small slopes, this is equivalent to dividing the change in elevation by the distance L measured along the channel bottom between two sections. For steady uniform flow, the velocity distribution is the same at all sections of the channel, so that the energy grade line has the same angle as the bottom of the channel, thus:

$$S = h_f/L$$

The distance between the liquid surface and the bottom of the channel is sometimes called the **stage** and is denoted by the symbol y in Fig. 3.3.33. When the stages between the sections are not uniform, that is, $y_1 \neq y_2$ or the cross section of the channel changes, or both, the flow is said to be **varied.** When a liquid flows in a channel of uniform cross section and the slope of the surface is the same as the slope of the bottom of the channel ($y_1 = y = y_2$), the flow is said to be **uniform.**

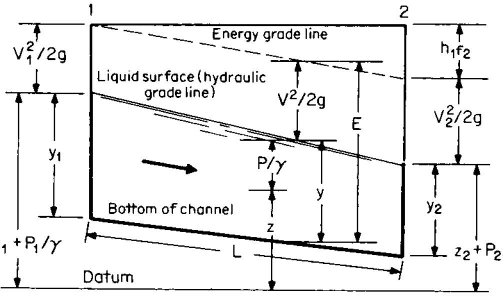


Fig. 3.3.33 Notation for open channel flow.

Parameters The forces acting on a liquid flowing in an open channel are inertia, viscous, surface tension, and gravity. If the channel has a surface roughness of ε, a hydraulic radius of R_h, and a slope of S, either similarity or dimensional analysis leads to $f(\mathbf{F}, \mathbf{W}, \mathbf{R}, \varepsilon/4R_h) = 0$, which may be written as $V = C\sqrt{R_h S}$, where $C = f(\mathbf{W}, \mathbf{R}, \varepsilon/4R_h)$ and is known as the **Chézy coefficient.** The relationship between the Chézy coefficient C and the friction factor may be determined by equating

$$V = \sqrt{8R_h h_f g/fL} = C\sqrt{R_h S} = C\sqrt{(R_h h_f)/L}$$

or $C = (8g/f)^{1/2}$. Although this establishes a relationship between the Chézy coefficient and the friction factor, it should be noted that $f = f(\mathbf{R}, \varepsilon/4R_h)$ and $C = f(\mathbf{W},\mathbf{R},\varepsilon/4R_h)$, because in open-channel flow, pressure forces are absent and in pipe flow, surface-tension and gravity forces are absent. For these reasons, data obtained in pipe flow should not be applied to open-channel flow.

Roughness Factors For open-channel flow, the Chézy coefficient is calculated by the Manning equation, which was developed from examination of experimental results of water tests. The **Manning relation** is stated as

$$C = \frac{1.486}{n} R_h^{1/6}$$

where n is a roughness factor and should be a function of Reynolds number, Weber number, and relative roughness. Since only water-test data obtained at ordinary temperatures support these values, it must be assumed that n is the value for turbulent flow only. Since surface tension is a weak property, the effects of Weber-number variation are negligible, leaving n to be some function of surface roughness. Design values of n are given in Table 3.3.17. Maximum flow for a given slope will take place when R_h is a maximum, and values of R_{hmax} are given in Table 3.3.6.

Table 3.3.17 Values of Roughness Factor n for Use in Manning Equation

Surface	n	Surface	n
Brick	0.015	Earth, with stones and weeds	0.035
Cast iron	0.015		
Concrete, finished	0.012	Gravel	0.029
Concrete, unfinished	0.015	Riveted steel	0.017
Brass pipe	0.010	Rubble	0.025
Earth	0.025	Wood, planed	0.012
		Wood, unplaned	0.013

SOURCE: Compiled from data given in R. Horton, *Engineering News,* 75, 373, 1916.

EXAMPLE. It is necessary to carry 150 ft³/s of water in a rectangular unplaned timber flume whose width is to be twice the depth of water. What are the required dimensions for various slopes of the flume? From Table 3.3.6, $A = b^2/2$ and $R_h = h/2 = b/4$. From Table 3.3.17, $n = 0.013$ for unplaned wood. From Manning's equation, $C = 1.486/n$, $R_h^{1/6} = (1.486/0.013)(b^{1/6}/(4)^{1/6} = 90.73\ b^{1/6}$. From the continuity equation, $V = Q/A = 150/(b^2/2)$, $V = 300/b^2$. From the Chézy equation, $V = C\sqrt{R_h S} = 300/b^2 = 90.73b^{1/6}\sqrt{(b/4)S}$; solving for b, $b = 2.0308/S^{3/16}$.

Assumed S:	1×10^{-1}	1×10^{-2}	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}	ft/ft
Required b:	3.127	4.816	7.416	11.42	17.59	27.08	ft

EXAMPLE. A rubble-lined trapezoidal canal with 45° sides is to carry 360 ft³/s of water at a depth of 4 ft. If the slope is 9×10^{-4} ft/ft, what should be the dimensions of the canal? From Table 3.3.17, $n = 0.025$ for rubble. From Table 3.3.6 for $\alpha = 45°$, $A = (b + h)h = 4(b + 4)$, and $R_h = (b + h)h/(b + 2.828h) = 4(b + 4)/(b + 11.312)$. From the Manning relation, $C = (1.486/n)(R_h^{1/6}) = (1.486/0.025)R_h^{1/6} = 59.44\ R_h^{1/6}$. For the first trial, assume $R_h = R_{hmax} = h/2 = 4/2 = 2$; then $C = 59.44(2)^{1/6} = 66.72$ and $V = C\sqrt{R_h S} = 66.72\sqrt{2 \times 9 \times 10^{-4}} = 2.831$. From the continuity equation, $A = Q/V = 360/2.831 = 127.2 = 4(b + 4)$; $b = 27.79$ ft. Second trial, use the first trial, $R_h = 4(27.79 + 4)/(27.79 + 11.312)$, $R_h = 3.252$, $V = 59.44(3.252)^{1/6}\sqrt{3.252 \times 9 \times 10^{-4}} = 3.914$. From the equation of continuity, $Q/V = 360/3.914 = 91.97 = 4(b + 4)$, $b = 18.99$. Subsequent trial-and-error solutions result in a balance at $b = 19.93$ ft (6.075 m).

Specific Energy Specific energy is defined as the energy of the fluid referred to the bottom of the channel as the datum. Thus the specific energy E at any section is given by $E = y + V^2/2g$; from the continuity equation $V = Q/A$ or $E = y + (Q/A)^2/2g$. For a rectangular channel whose width is b, $A = by$; and if $\mathbf{q}$ is defined as the flow rate per unit width, $\mathbf{q} = Q/b$ and $E = y + (\mathbf{q}b/by)^2/2g = y + (\mathbf{q}/y)^2/2g$.

Critical Values For rectangular channels, if the specific-energy equation is differentiated and set equal to zero, critical values are obtained; thus $dE/dy = d/dy\,[y + (\mathbf{q}/y)^2/2g] = 0 = 1 - \mathbf{q}^2/y^3g$ or $\mathbf{q}_c^2 = y_c^3g$. Substituting in the specific-energy equation, $E = y_c + y_c^3g/2gy_c^2 = 3/2y_c$. Figure 3.3.34 shows the relation between depth and specific energy for a constant flow rate. If the depth is greater than critical, the flow is *subcritical;* at critical depth it is *critical* and at depths below critical the flow is *supercritical.* For a given specific energy, there is a maximum unit flow rate that can exist.

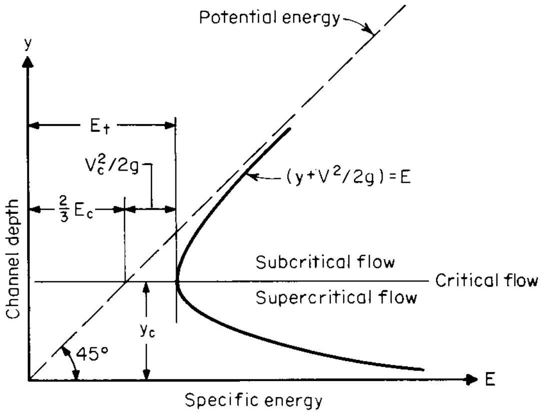


Fig. 3.3.34 Specific energy diagram, constant flow rate.

The **Froude number** $\mathbf{F} = V/\sqrt{gy}$, when substituted in the specific-energy equation, yields $E = y + (\mathbf{F}^2gy)/2g = y(1 + \mathbf{F}^2/2)$ or $E/y = 1 + \mathbf{F}^2/2$. For critical flow, $E_c/y_c = 3/2$. Substituting $E_c/y_c = 3/2 = 1 + \mathbf{F}_c^2/2$, or $\mathbf{F} = 1$,

$\mathbf{F} < 1$ Flow is subcritical
$\mathbf{F} = 1$ Flow is critical
$\mathbf{F} > 1$ Flow is supercritical

It is seen that for open-channel flow the Froude number determines the type of flow in the same manner as Mach number for compressible flow.

EXAMPLE. Water flows at a ate of 600 ft³/s in a rectangular channel 10 ft wide at a depth of 4 ft. Determine (1) specific energy and (2) type of flow.

1. from the continuity equation,

$$V = Q/A = 600/(10 \times 4) = 15 \text{ ft/s}$$
$$E = y + V^2/2g = 4 + (15)^2/2(2 \times 32.17) = 7.497 \text{ ft}$$

2. $\mathbf{F} = V/\sqrt{gy} = 15/\sqrt{32.17 \times 4} = 1.322$; $\mathbf{F} > 1$ ∴ flow is supercritical.

FLOW OF LIQUIDS FROM TANK OPENINGS

Steady State Consider the jet whose velocity is V discharging from an open tank through an opening whose area is a, as shown in Fig. 3.3.35. The liquid height above the centerline is h, and the cross-sectional area of the tank at h is A. The ideal velocity of the jet is $V_i = \sqrt{2gh}$. The ratio of the actual velocity V to the ideal velocity V_i is the **coefficient of velocity** Cv, or $V = C_vV_i = C_v\sqrt{2gh}$. The ratio of the actual opening a to the minimum area of the jet a_c is the **coefficient of contraction** C_c, or $a = C_ca_c$. The ratio of the actual discharge Q to the ideal discharge Q_i is the **coefficient of discharge** C, or $Q = CQ_i = C_aV_i = C_cC_va\sqrt{2gh}$, and $C = C_cC_v$. Nominal values of coefficients for various openings are given in Fig. 3.3.36.

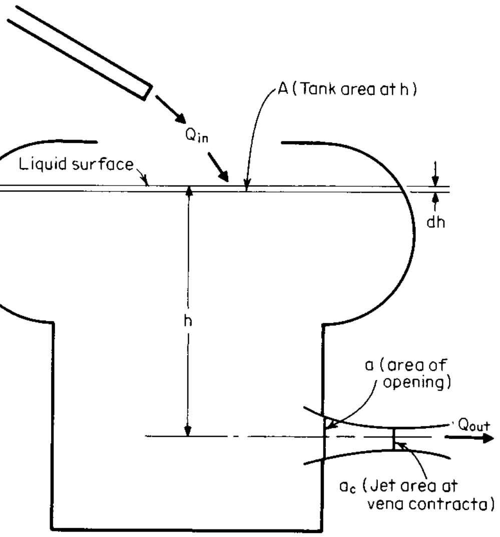


Fig. 3.3.35 Notation for tank flow.

Unsteady State If the rate of liquid entering the tank Q_{in} is different from that leaving, the level h in the tank will change because of the change in storage. For liquids, the conservation-of-mass equation may be written as $Q_{in} - Q_{out} = Q_{stored}$; for a time interval dt, $(Q_{in} - Q_{out})dt =$

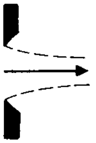
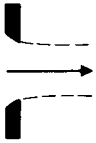
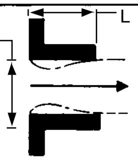
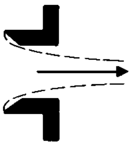
Type	Coefficient		
	C	Cc	Cv
Sharp-edged orifice 	0.61	0.62	0.98
Rounded-edged orifice 	0.98	1.00	0.98
Short tube 	0.80	1.00	0.80
Borda 	0.51	0.52	0.98

Fig. 3.3.36 Nominal coefficients of orifices.

$A\,dh$, neglecting fluid acceleration,

$$\begin{aligned}Q_{out}\,dt &= Ca\sqrt{2gh}\,dt, \text{ or } (Q_{in} - Ca\sqrt{2gh})\,dt \\ &= A\,dh, \text{ or } \int_{t_1}^{t_2} dt = \int_{h_1}^{h_2} \frac{A\,dh}{Q_{in} - Q_{out}} \\ &= \int_{h_1}^{h_2} \frac{A\,dh}{Q_{in} - Ca\sqrt{2gh}}\end{aligned}$$

Example. An open cylindrical tank is 6 ft in diameter and is filled with water to a depth of 10 ft. A 4-in-diameter sharp-edged orifice is installed on the bottom of the tank. A pipe on the top of the tank supplies water at the rate of 1 ft³/s. Estimate (1) the steady-state level of this tank, (2) the time required to reduce the tank level by 2 ft.

1. *Steady-state level*. From Fig. 3.3.36, $C = 0.61$ for a sharp-edged orifice, $a = (\pi/4)d^2 = (\pi/4)(4/12)^2 = 0.08727$ ft². For steady state, $Q_{in} = Q_{out} = Ca\sqrt{2gh} = 1 = (0.61)(0.08727)(2 \times 32.17h)^{1/2}$; $h = 5.484$ ft.
2. Time required to lower level 2 ft, $A = (\pi/4)D^2 = (\pi/4)(6)^2 = 28.27$ ft²

$$t_2 - t_1 = \int_{h_1}^{h_2} \frac{A\,dh}{Q_{in} - Ca\sqrt{2gh}}$$

This equation may be integrated by letting $Q = Ca\sqrt{2g}\,h^{1/2}$; then $dh = 2Q\,dQ/(Ca\sqrt{2g})^2$; then

$$t_2 - t_1 = \frac{2A}{(Ca\sqrt{2g})^2}\left[Q_{in}\log_e\left(\frac{Q_{in} - Q_1}{Q_{in} - Q_2}\right) + Q_1 - Q_2\right]$$

At t_1: $Q_1 = 0.61 \times 0.08727\sqrt{2 \times 32.17 \times 10} = 1.350$ ft³/s
At t_2: $Q_2 = 0.61 \times 0.08727\sqrt{2 \times 32.17 \times 8} = 1.208$ ft³/s

$$t_2 - t_1 = \frac{2 \times 28.27}{(0.61 \times 0.08727\sqrt{2 \times 32.17})^2} \times \left[(1)\log_e\left(\frac{1 - 1.350}{1 - 1.208}\right) + 1.350 - 1.208\right]$$

$$t_2 - t_1 = 205.4 \text{ s}$$

WATER HAMMER

Equations Water hammer is the series of shocks, sounding like hammer blows, produced **by suddenly reducing the flow** of a fluid in a pipe. Consider a fluid flowing frictionlessly in a rigid pipe of uniform area A with a velocity V. The pipe has a length L, and inlet pressure p_1 and a pressure p_2 at L. At length L, there is a valve which can suddenly reduce the velocity at L to $V - \Delta V$. The equivalent mass rate of flow of a pressure wave traveling at sonic velocity c, $\dot{M} = \rho Ac$. From the impulse-momentum equation, $\dot{M}(V_2 - V_1) = p_2A_2 - p_1A_1$; for this application, $(\rho Ac)(V - \Delta V - V) = p_2A - p_1A$, or the increase in pressure $\Delta p = -\rho c\Delta V$. When the liquid is flowing in an elastic pipe, the equation for pressure rise must be modified to account for the expansion of the pipe; thus

$$c = \sqrt{\frac{E_s}{\rho[1 + (E_s/E_p)(D_o + D_i)/(D_o - D_i)]}}$$

where ρ = mass density of the fluid, E_s = bulk modulus of elasticity of the fluid, E_p = modulus of elasticity of the pipe material, D_o = outside diameter of pipe, and D_i = inside diameter of pipe.

Time of Closure The time for a pressure wave to travel the length of pipe L and return is $t = 2L/c$. If the time of closure $t_c \le t$, the approximate pressure rise $\Delta p \approx -2\,\rho V(L/t_c)$. When it is not feasible to close the valve slowly, **air chambers** or **surge tanks** may be used to absorb all or most of the pressure rise. Water hammer can be very dangerous. See Sec. 9.9.

Example. Water flows at 68°F (20°C) in a 3-in steel schedule 40 pipe at a velocity of 10 ft/s. A valve located 200 ft downstream is suddenly closed. Determine (1) the increase in pressure considering pipe to be rigid, (2) the increase considering pipe to be elastic, and (3) the maximum time of valve closure to be considered ''sudden.''

For water, $\rho = -1.937$ slugs/ft³ = 1.937 lb · sec²/ft⁴; $E_s = 319{,}000$ lb/in²; $E_p = 28.5 \times 10^6$ lb/in² (Secs. 5.1 and 6); $c = 4{,}860$ ft/s; from Sec. 8.7, $D_o = 3.5$ in, $D_i = 3.068$ in.

1. *Inelastic pipe*

$$\begin{aligned}\Delta p &= -\rho c\Delta V = -(1.937)(4{,}860)(-10) = 94{,}138 \text{ lbf/ft}^2 \\ &= 94{,}138/144 = 653.8 \text{ lbf/in}^2\ (4.507 \times 10^6 \text{ N/m}^2)\end{aligned}$$

2. *Elastic pipe*

$$\begin{aligned}c &= \sqrt{\frac{E_s}{\rho[1 + (E_s/E_p)(D_o + D_i)/(D_o - D_i)]}} \\ &= \sqrt{\frac{319{,}000 \times 144}{1.937\left[1 + \dfrac{(319{,}000/28.5 \times 10^6)(3.500 + 3.067)}{(3.500 - 3.067)}\right]}} \\ &= 4{,}504\end{aligned}$$

$$\begin{aligned}\Delta p &= -(1.937)(4{,}504)(-10) \\ &= 87{,}242 \text{ lbf/ft}^2 = 605.9 \text{ lbf/in}^2\ (4.177 \times 10^6 \text{ N/m}^2)\end{aligned}$$

3. *Maximum time for closure*

$$t = 2L/c = 2 \times 200/4{,}860 = 0.08230 \text{ s or less than 1/10 s}$$

3.4 Vibration

by Leonard Meirovitch

References: Harris, ''Shock and Vibration Handbook,'' 3d ed., McGraw-Hill. Thomson, ''Theory of Vibration with Applications,'' 4th ed., Prentice Hall. Meirovitch, ''Elements of Vibration Analysis,'' 2d ed., McGraw-Hill. Meirovitch, ''Principles and Techniques of Vibrations,'' Prentice-Hall.

SINGLE-DEGREE-OF-FREEDOM SYSTEMS

Discrete System Components A **system** is defined as an aggregation of components acting together as one entity. The components of a vibratory mechanical system are of three different types, and they relate forces to displacements, velocities, and accelerations. The component relating forces to displacements is known as a **spring** (Fig. 3.4.1*a*). For a **linear spring** the force F_s is proportional to the elongation $\delta = x_2 - x_1$, or

$$F_s = k\delta = k(x_2 - x_1) \qquad (3.4.1)$$

where k represents the **spring constant,** or the **spring stiffness,** and x_1 and x_2 are the displacements of the end points. The component relating forces to velocities is called a **viscous damper** or a **dashpot** (Fig. 3.4.1*b*). It consists of a piston fitting loosely in a cylinder filled with liquid so that the liquid can flow around the piston when it moves relative to the cylinder. The relation between the damper force and the velocity of the piston relative to the cylinder is

$$F_d = c(\dot{x}_2 - \dot{x}_1) \qquad (3.4.2)$$

in which c is the **coefficient of viscous damping;** note that dots denote derivatives with respect to time. Finally, the relation between forces and accelerations is given by Newton's second law of motion:

$$F_m = m\ddot{x} \qquad (3.4.3)$$

where m is the **mass** (Fig. 3.4.1*c*).

The spring constant k, coefficient of viscous damping c, and mass m represent physical properties of the components and are the **system parameters**. By implication, these properties are concentrated at points, thus they are **lumped,** or **discrete, parameters**. Note that springs and dampers are assumed to be massless and masses are assumed to be rigid.

Springs can be arranged in parallel and in series. Then, the proportionality constant between the forces and the end points is known as an **equivalent spring constant** and is denoted by k_{eq}, as shown in Table 3.4.1. Certain elastic components, although distributed over a given line segment, can be regarded as lumped with an equivalent spring constant given by $k_{\text{eq}} = F/\delta$, where δ is the deflection at the point of application of the force F. A similar relation can be given for springs in torsion. Table 3.4.1 lists the equivalent spring constants for a variety of components.

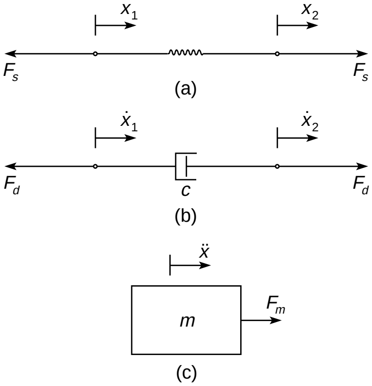


Fig. 3.4.1

Equation of Motion The dynamic behavior of many engineering systems can be approximated with good accuracy by the **mass-damper-spring model** shown in Fig. 3.4.2. Using Newton's second law in conjunction with Eqs. (3.4.1) to (3.4.3) and measuring the displacement $x(t)$ from the static equilibrium position, we obtain the differential equation of motion

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \qquad (3.4.4)$$

which is subject to the **initial conditions** $x(0) = x_0$, $\dot{x}(0) = v_0$, where x_0 and v_0 are the **initial displacement** and **initial velocity,** respectively. Equation (3.4.4) is in terms of a single coordinate, namely $x(t)$; the system of Fig. 3.4.2 is therefore said to be a **single-degree-of-freedom system.**

Free Vibration of Undamped Systems Assuming zero damping and external forces and dividing Eq. (3.4.4) through by m, we obtain

$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{k/m} \qquad (3.4.5)$$

In this case, the vibration is caused by the initial excitations alone. The solution of Eq. (3.4.5) is

$$x(t) = A \cos(\omega_n t - \phi) \qquad (3.4.6)$$

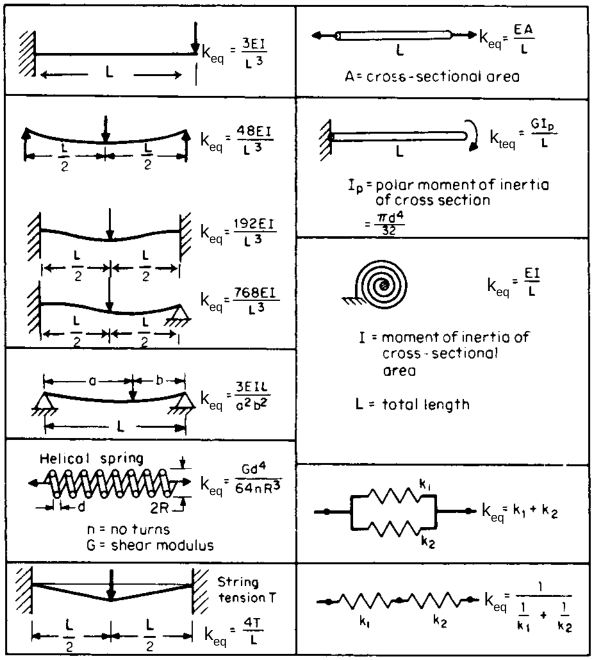
which represents **simple sinusoidal,** or **simple harmonic oscillation** with **amplitude** A, **phase angle** ϕ, and **frequency**

$$\omega_n = \sqrt{k/m} \qquad \text{rad/s} \qquad (3.4.7)$$

Systems described by equations of the type (3.4.5) are called **harmonic oscillators.** Because the frequency of oscillation represents an inherent property of the system, independent of the initial excitation, ω_n is called the **natural frequency**. On the other hand, the amplitude and phase angle do depend on the initial displacement and velocity, as follows:

$$A = \sqrt{x_0^2 + (v_0/\omega_n)^2} \qquad \phi = \tan^{-1} v_0/x_0\omega_n \qquad (3.4.8)$$

Table 3.4.1 Equivalent Spring Constants



$k_{\text{eq}} = \frac{3EI}{L^3}$	$k_{\text{eq}} = \frac{EA}{L}$ A = cross-sectional area
$k_{\text{eq}} = \frac{48EI}{L^3}$ $k_{\text{eq}} = \frac{192EI}{L^3}$ $k_{\text{eq}} = \frac{768EI}{L^3}$	$k_{\text{teq}} = \frac{GI_p}{L}$ I_p = polar moment of inertia of cross section $= \frac{\pi d^4}{32}$
$k_{\text{eq}} = \frac{3EIL}{a^2b^2}$	$k_{\text{eq}} = \frac{EI}{L}$ I = moment of inertia of cross-sectional area L = total length
Helical spring $k_{\text{eq}} = \frac{Gd^4}{64nR^3}$ n = no turns G = shear modulus	$k_{\text{eq}} = k_1 + k_2$
String tension T $k_{\text{eq}} = \frac{4T}{L}$	$k_{\text{eq}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$

The time necessary to complete one cycle of motion defines the **period**

$$T = 2\pi/\omega_n \qquad \text{seconds} \qquad (3.4.9)$$

The reciprocal of the period provides another definition of the **natural frequency,** namely,

$$f_n = \frac{1}{T} = \frac{\omega_n}{2\pi} \qquad \text{Hz} \qquad (3.4.10)$$

where Hz denotes *hertz* [1 Hz = 1 cycle per second (cps)].

A large variety of vibratory systems behave like harmonic oscillators, many of them when restricted to small amplitudes. Table 3.4.2 shows a variety of harmonic oscillators together with their respective natural frequency.

Free Vibration of Damped Systems Let $F(t) = 0$ and divide through by m. Then, Eq. (3.4.4) reduces to

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0 \qquad (3.4.11)$$

where

$$\zeta = c/2m\omega_n \qquad (3.4.12)$$

is the **damping factor**, a nondimensional quantity. The nature of the motion depends on ζ. The most important case is that in which $0 < \zeta < 1$.

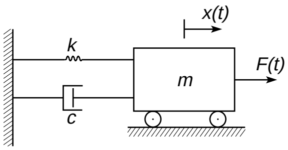
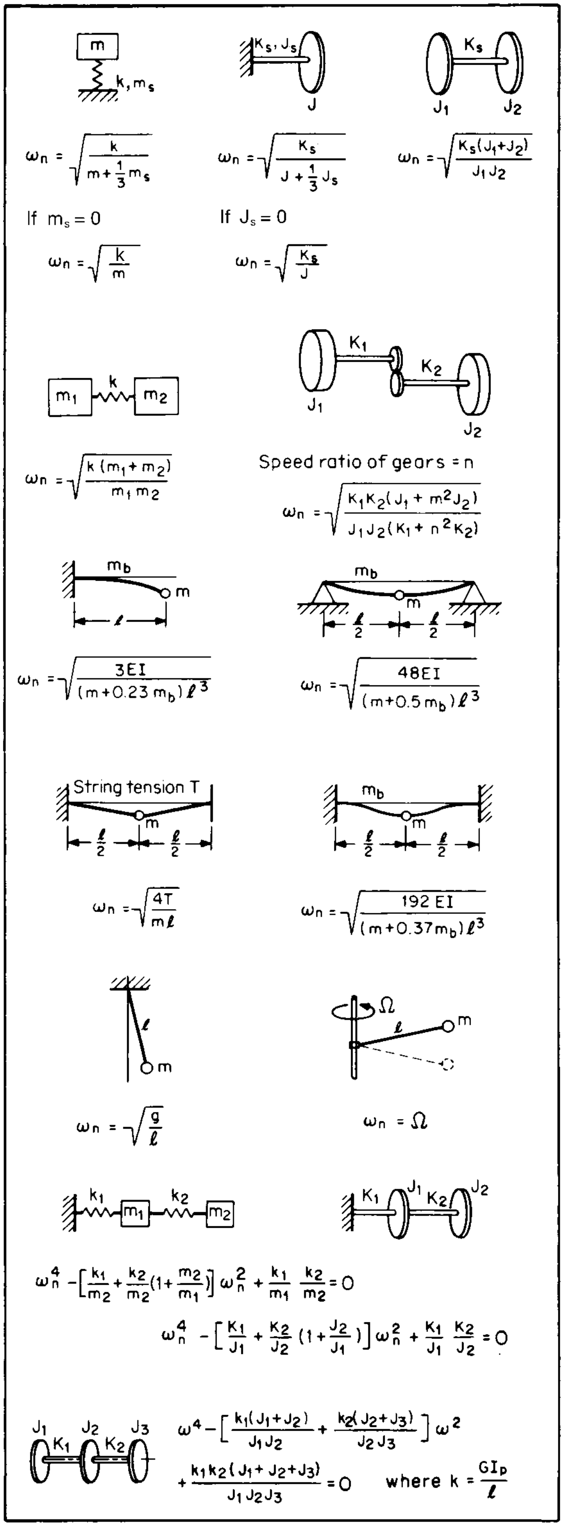


Fig. 3.4.2

Table 3.4.2 Harmonic Oscillators and Natural Frequencies



$$\omega_n = \sqrt{\frac{k}{m + \frac{1}{3} m_s}} \qquad \omega_n = \sqrt{\frac{K_s}{J + \frac{1}{3} J_s}} \qquad \omega_n = \sqrt{\frac{K_s (J_1 + J_2)}{J_1 J_2}}$$

If $m_s = 0$ $\qquad$ If $J_s = 0$

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \omega_n = \sqrt{\frac{K_s}{J}}$$

Speed ratio of gears = n

$$\omega_n = \sqrt{\frac{k (m_1 + m_2)}{m_1 m_2}} \qquad \omega_n = \sqrt{\frac{K_1 K_2 (J_1 + m^2 J_2)}{J_1 J_2 (K_1 + n^2 K_2)}}$$

$$\omega_n = \sqrt{\frac{3EI}{(m + 0.23 m_b) \ell^3}} \qquad \omega_n = \sqrt{\frac{48EI}{(m + 0.5 m_b) \ell^3}}$$

$$\omega_n = \sqrt{\frac{4T}{m\ell}} \qquad \omega_n = \sqrt{\frac{192\,EI}{(m + 0.37 m_b) \ell^3}}$$

$$\omega_n = \sqrt{\frac{g}{\ell}} \qquad \omega_n = \Omega$$

$$\omega_n^4 - \left[\frac{k_1}{m_2} + \frac{k_2}{m_2}\left(1 + \frac{m_2}{m_1}\right)\right] \omega_n^2 + \frac{k_1}{m_1} \frac{k_2}{m_2} = 0$$

$$\omega_n^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_2}\left(1 + \frac{J_2}{J_1}\right)\right] \omega_n^2 + \frac{K_1}{J_1} \frac{K_2}{J_2} = 0$$

$$\omega^4 - \left[\frac{k_1 (J_1 + J_2)}{J_1 J_2} + \frac{k_2 (J_2 + J_3)}{J_2 J_3}\right] \omega^2 + \frac{k_1 k_2 (J_1 + J_2 + J_3)}{J_1 J_2 J_3} = 0 \qquad \text{where } k = \frac{GI_p}{\ell}$$

In this case, the system is said to be **underdamped** and the solution of Eq. (3.4.11) is

$$x(t) = A e^{-\zeta \omega_n t} \cos(\omega_d t - \phi) \tag{3.4.13}$$

where

$$\omega_d = (1 - \zeta^2)^{1/2} \omega_n \tag{3.4.14}$$

is the **frequency of damped free vibration** and

$$T = 2\pi / \omega_d \tag{3.4.15}$$

is the **period of damped oscillation.** The amplitude and phase angle depend on the initial displacement and velocity, as follows:

$$A = \sqrt{x_0^2 + (\zeta \omega_n x_0 + v_0)^2 / \omega_d^2} \qquad \phi = \tan^{-1} (\zeta \omega_n x_0 + v_0) / x_0 \omega_d \tag{3.4.16}$$

The motion described by Eq. (3.4.13) represents **decaying oscillation,** where the term $A e^{-\zeta \omega_n t}$ can be regarded as a time-dependent amplitude, providing an envelope bounding the harmonic oscillation.

When $\zeta \geq 1$, the solution represents **aperiodic decay.** The case $\zeta = 1$ represents **critical damping,** and

$$c_c = 2 m \omega_n \tag{3.4.17}$$

is the **critical damping coefficient,** although there is nothing critical about it. It merely represents the borderline between oscillatory decay and aperiodic decay. In fact, c_c is the smallest damping coefficient for which the motion is aperiodic. When $\zeta > 1$, the system is said to be **overdamped.**

Logarithmic Decrement Quite often the damping factor is not known and must be determined experimentally. In the case in which the system is underdamped, this can be done conveniently by plotting $x(t)$ versus t (Fig. 3.4.3) and measuring the response at two different times separated by a complete period. Let the times be t_1 and $t_1 + T$, introduce the notation $x(t_1) = x_1$, $x(t_1 + T) = x_2$, and use Eq. (3.4.13) to obtain

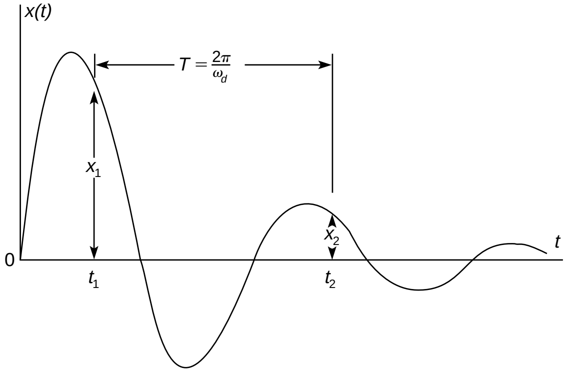


Fig. 3.4.3

$$\frac{x_1}{x_2} = \frac{A e^{-\zeta \omega_n t_1} \cos (\omega_d t_1 - \phi)}{A e^{-\zeta \omega_n (t_1 + T)} \cos [\omega_d (t_1 + T) - \phi]} = e^{\zeta \omega_n T} \tag{3.4.18}$$

where $\cos [\omega_d (t_1 + T) - \phi] = \cos (\omega_d t_1 - \phi + 2\pi) = \cos (\omega_d t_1 - \phi)$. Equation (3.4.18) yields the **logarithmic decrement**

$$\delta = \ln \frac{x_1}{x_2} = \zeta \omega_n T = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} \tag{3.4.19}$$

which can be used to obtain the damping factor

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{3.4.20}$$

For small damping, the logarithmic decrement is also small, and the damping factor can be approximated by

$$\zeta \approx \frac{\delta}{2\pi} \tag{3.4.21}$$

Response to Harmonic Excitations Consider the case in which the excitation force $F(t)$ in Eq. (3.4.4) is harmonic. For convenience, express $F(t)$ in the form $kA \cos \omega t$, where k is the spring constant, A is an amplitude with units of displacement and ω is the **excitation frequency**. When divided through by m, Eq. (3.4.4) has the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 A \cos \omega t \qquad (3.4.22)$$

The solution of Eq. (3.4.22) can be expressed as

$$x(t) = A|G(\omega)| \cos (\omega t - \phi) \qquad (3.4.23)$$

where

$$|G(\omega)| = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \qquad (3.4.24)$$

is a **nondimensional magnitude factor*** and

$$\phi(\omega) = \tan^{-1} \frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2} \qquad (3.4.25)$$

is the phase angle; note that both the magnitude factor and phase angle depend on the excitation frequency ω.

Equation (3.4.23) shows that the response to harmonic excitation is also harmonic and has the same frequency as the excitation, but different amplitude $A|G(\omega)|$ and phase angle $\phi(\omega)$. Not much can be learned by plotting the response as a function of time, but a great deal of information can be gained by plotting $|G|$ versus ω/ω_n and ϕ versus ω/ω_n. They are shown in Fig. 3.4.4 for various values of the damping factor ζ.

In Fig. 3.4.4, for low values of ω/ω_n, the nondimensional magnitude factor $|G(\omega)|$ approaches unity and the phase angle $\phi(\omega)$ approaches zero. For large values of ω/ω_n, the magnitude approaches zero (see accompanying footnote about magnification factor) and the phase angle approaches 180°. The magnitude experiences peaks for $\omega/\omega_n = \sqrt{1 - 2\zeta^2}$, provided $\zeta < 1/\sqrt{2}$. The peak values are $|G(\omega)|_{\max} = 1/2\zeta\sqrt{1 - \zeta^2}$. For small ζ, the peaks occur approximately at $\omega/\omega_n = 1$ and have the approximate values $|G(\omega)|_{\max} = Q \approx 1/2\zeta$, where Q is known as the **quality factor**. In such cases, the phase angle tends to 90°. Clearly, for small ζ the system experiences large-amplitude vibration, a condition known as **resonance**. The points P_1 and P_2, where $|G|$ falls to $Q/\sqrt{2}$, are called **half-power points**. The increment of frequency associated with the half-power points P_1 and P_2 represents the **bandwidth** $\Delta\omega$ of the system. For small damping, it has the value

$$\Delta\omega = \omega_2 - \omega_1 \approx 2\zeta\omega_n \qquad (3.4.26)$$

The case $\zeta = 0$ deserves special attention. In this case, referring to Eq. (3.4.22), the response is simply

$$x(t) = \frac{A}{1 - (\omega/\omega_n)^2} \cos \omega t \qquad (3.4.27)$$

For $\omega/\omega_n < 1$, the displacement is in the same direction as the force, so that the phase angle is zero; the **response is in phase** with the excitation. For $\omega/\omega_n > 1$, the displacement is in the direction opposite to the force, so that the phase angle is 180° **out of phase** with the excitation. Finally, when $\omega = \omega_n$ the response is

$$x(t) = \frac{A}{2} \omega_n t \sin \omega_n t \qquad (3.4.28)$$

This is typical of the **resonance** condition, when the response increases without bounds as time increases. Of course, at a certain time the displacement becomes so large that the spring ceases to be linear, thus violating the original assumption and invalidating the solution. In practical terms, unless the excitation frequency varies, passing quickly through $\omega = \omega_n$, the system can break down.

When the excitation is $F(t) = kA \sin \omega t$, the response is

$$x(t) = A|G(\omega)| \sin (\omega t - \phi) \qquad (3.4.29)$$

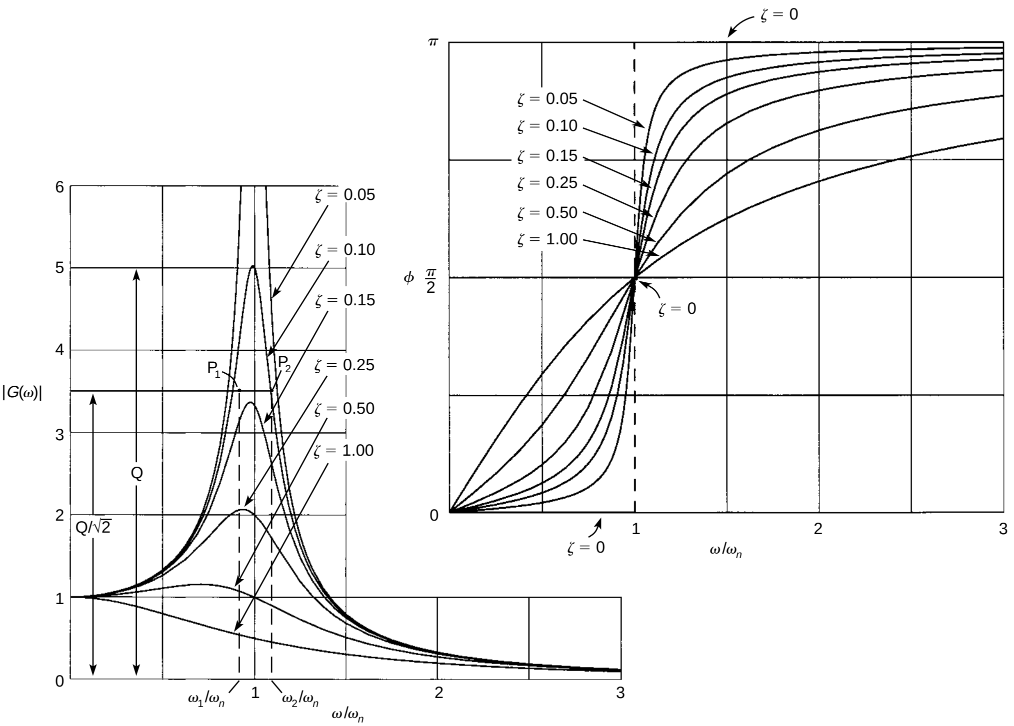


Fig. 3.4.4 Frequency response plots.

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* The term $|G(\omega)|$ is often referred to as *magnification factor*, but this is a misnomer, as we shall see shortly.

One concludes that in **harmonic response,** time plays a secondary role to the frequency. In fact, the only significant information is extracted from the magnitude and phase angle plots of Fig. 3.4.4. They are referred to as **frequency-response plots**.

Since time plays no particular role, the harmonic response is called **steady-state response**. In general, for linear systems with constant parameters, such as the mass-damper-spring system under consideration, the response to the initial excitations is added to the response to the excitation forces. The response to initial excitations, however, represents **transient response**. This is due to the fact that every system possesses some amount of damping, so that the response to initial excitations disappears with time. In contrast, steady-state response persists with time. Hence, in the case of harmonic excitations, it is meaningless to add the response to initial excitations to the harmonic response.

Vibration Isolation A problem of great interest is the magnitude of the force transmitted to the base by a system of the type shown in Fig. 3.4.2 subjected to harmonic excitation. This force is a combination of the spring force kx and the dashpot force $c\dot{x}$. Recalling Eq. (3.4.23), write

$$\begin{aligned} kx &= kA|G|\cos(\omega t - \phi) \\ c\dot{x} &= -c\omega A|G|\sin(\omega t - \phi) \\ &= c\omega A|G|\cos\left(\omega t - \phi + \frac{\pi}{2}\right) \end{aligned} \tag{3.4.30}$$

so that the dashpot force is 90° out of phase with the spring force. Hence, the magnitude of the force is

$$\begin{aligned} F_{\text{tr}} &= \sqrt{(kA|G|)^2 + (c\omega A|G|)^2} = kA|G|\sqrt{1 + (c\omega/k)^2} \\ &= kA|G|\sqrt{1 + (2\zeta\omega/\omega_n)^2} \end{aligned} \tag{3.4.31}$$

Let the magnitude of the harmonic excitation be $F_0 = kA$; the force transmitted to the base is then

$$\begin{aligned} T &= \frac{F_{\text{tr}}}{F_0} = |G|\sqrt{1 + (2\zeta\omega/\omega_n)^2} \\ &= \sqrt{\frac{1 + (2\zeta\omega/\omega n)^2}{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \end{aligned} \tag{3.4.32}$$

which represents a nondimensional ratio called **transmissibility**. Figure 3.4.5 plots F_{tr}/F_0 versus ω/ω_n for various values of ζ.

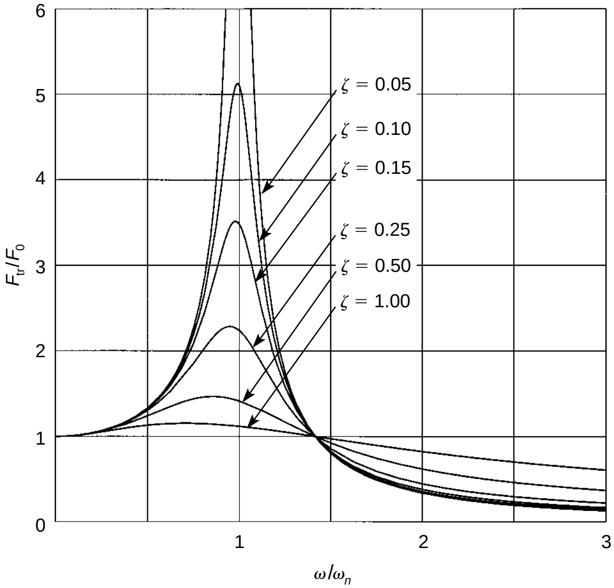


Fig. 3.4.5

The transmissibility is less than 1 for $\omega/\omega_n > \sqrt{2}$, and decreases as ω/ω_n increases. Hence, for **an isolator to perform well,** its natural frequency must be much smaller than the excitation frequency. However, for very low natural frequencies, difficulties can be encountered in isolator design. Indeed, the natural frequency is related to the static deflection δ_{st} by $\omega_n = \sqrt{k/m} = \sqrt{g/\delta_{\text{st}}}$, where g is the gravitational constant. For the natural frequency to be sufficiently small, the static deflection may have to be impractically large. The relation between the excitation frequency f measured in rotations per minute and the static deflection δ_{st} measured in inches is

$$f = 187.7\sqrt{\frac{2 - R}{\delta_{\text{st}}(1 - R)}} \qquad \text{rpm} \tag{3.4.33}$$

where $R = 1 - T$ **represents the percent reduction** in vibration. Figure 3.4.6 shows a logarithmic plot of f versus δ_{st} with R as a parameter.

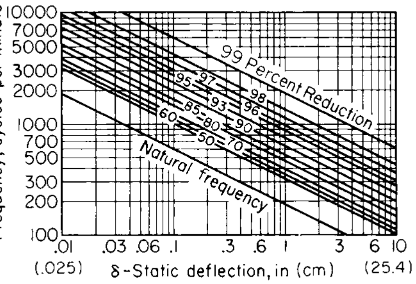


Fig. 3.4.6

Figure 3.4.7 depicts two types of isolators. In Fig. 3.4.7a, isolation is accomplished by means of springs and in Fig. 3.4.7b by rubber rings supporting the bearings. Isolators of all shapes and sizes are available commercially.

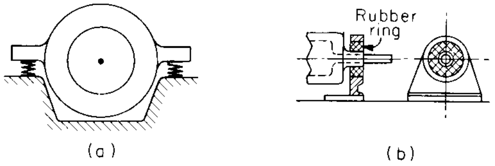


Fig. 3.4.7

Rotating Unbalanced Masses Many appliances, machines, etc., involve components spinning relative to a main body. A typical example is the clothes dryer. Under certain circumstances, the mass of the spinning component is not symmetric relative to the center of rotation, as when the clothes are not spread uniformly in the spinning drum, giving rise to harmonic excitation. The behavior of such systems can be simulated adequately by the single-degree-of-freedom model shown in Fig. 3.4.8, which consists of a main mass $M - m$, supported by two springs of combined stiffness k and a dashpot with coefficient of viscous damping c, and two eccentric masses $m/2$ rotating in opposite sense with the constant angular velocity ω. Although there are three masses, the motion of the eccentric masses relative to the main mass is prescribed, so that there is only one degree of freedom. The equation of motion for the system is

$$M\ddot{x} + c\dot{x} + kx = ml\omega^2 \sin \omega t \tag{3.4.34}$$

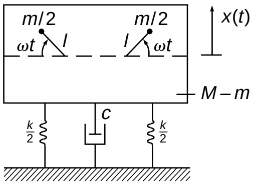


Fig. 3.4.8

Using the analogy with Eq. (3.4.29), the solution of Eq. (3.4.34) is

$$x(t) = \frac{m}{M}\, l \left(\frac{\omega}{\omega_n}\right)^2 |G(\omega)| \sin(\omega t - \phi) \qquad \omega_n^2 = \frac{k}{M} \tag{3.4.35}$$

The magnitude factor in this case is $(\omega/\omega_n)^2|G(\omega)|$, where $|G(\omega)|$ is given by Eq. (3.4.24); it is plotted in Fig. 3.4.9. On the other hand, the phase angle remains as in Fig. 3.4.4.

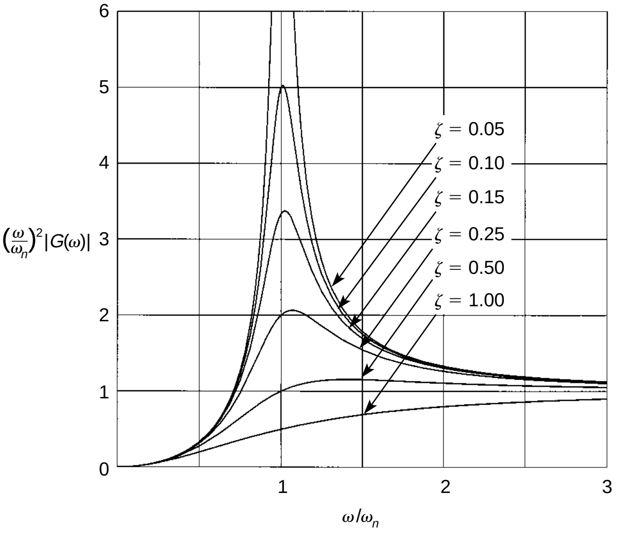
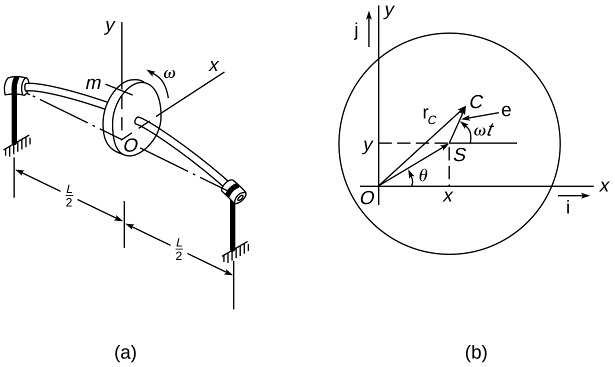


Fig. 3.4.9

Whirling of Rotating Shafts Many mechanical systems involve rotating shafts carrying disks. If the disk has some eccentricity, then the centrifugal forces cause the shaft to bend, as shown in Fig. 3.4.10*a*. The rotation of the plane containing the bent shaft about the bearing axis is called **whirling**. Figure 3.4.10*b* shows a disk with the body axes *x,y* rotating about the origin *O* with the angular velocity ω. The geometrical center of the disk is denoted by *S* and the mass center by *C*. The distance between the two points is the eccentricity *e*. The shaft is massless and of stiffness k_{eq} and the disk is rigid and of mass *m*. The *x* and *y* components of the displacement of *S* relative to *O* are independent from one another and, for no damping, satisfy the equations of motion



(a) (b)

Fig. 3.4.10

$$\ddot{x} + \omega_n^2 x = e\omega^2 \cos \omega t \qquad \ddot{y} + \omega_n^2 y = e\omega^2 \sin \omega t \qquad \omega_n^2 = k_{\text{eq}}/m \tag{3.4.36}$$

where, assuming that the shaft is simply supported (see Table 3.4.1), $k_{\text{eq}} = 48EI/L^3$, in which *E* is the modulus of elasticity, *I* the cross-sectional area moment of inertia, and *L* the length of the shaft. By analogy with Eq. (3.4.27), Eqs. (3.4.36) have the solution

$$x(t) = \frac{e(\omega/\omega_n)^2}{1 - (\omega/\omega_n)^2} \cos \omega t \qquad y(t) = \frac{e(\omega/\omega_n)^2}{1 - (\omega/\omega_n)^2} \sin \omega t \tag{3.4.37}$$

Clearly, resonance occurs when the whirling angular velocity coincides with the natural frequency. In terms of rotations per minute, it has the value

$$f_c = \frac{60}{2\pi}\, \omega_n = \frac{60}{2\pi}\sqrt{\frac{48EI}{mL^3}} \qquad \text{rpm} \tag{3.4.38}$$

where f_c is called the **critical speed**.

Structural Damping Experience shows that energy is dissipated in all real systems, including those assumed to be undamped. For example, because of internal friction, energy is dissipated in real springs undergoing cyclic stress. This type of damping is called **structural damping** or **hysteretic damping** because the energy dissipated in one cycle of stress is equal to the area inside the hysteresis loop. Systems possessing structural damping and subjected to harmonic excitation with the frequency ω can be treated as if they possess viscous damping with the equivalent coefficient

$$c_{\text{eq}} = \alpha/\pi\omega \tag{3.4.39}$$

where α is a material constant. In this case, the equation of motion is

$$m\ddot{x} + \frac{\alpha}{\pi\omega}\dot{x} + kx = kA \cos \omega t \tag{3.4.40}$$

The solution of Eq. (3.4.40) is

$$x(t) = A|G| \cos(\omega t - \phi) \tag{3.4.41}$$

where this time the magnitude factor and phase angle have the values

$$G = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + \gamma^2}} \qquad \phi = \tan^{-1} \frac{\gamma\omega_n^2}{\omega[1 - (\omega/\omega_n)^2]} \tag{3.4.42}$$

in which

$$\gamma = \frac{\alpha}{\pi k} \tag{3.4.43}$$

is known as the **structural damping factor**. One word of caution is in order: **the analogy between structural and viscous damping is valid only for harmonic excitation.**

Balancing of Rotating Machines Machines such as electric motors and generators, turbines, compressors, etc. contain rotors with journals supported by bearings. In many cases, the rotors rotate relative to the bearings at very high rates, reaching into tens of thousands of revolutions per minute. Ideally the rotor is rigid and the axis of rotation coincides with one of its principal axes; by implication, the rotor center of mass lies on the axis of rotation. Such a rotor does not wobble and the only forces exerted on the bearings are due to the weight of the rotor. Such a rotor is said to be **perfectly balanced**. These ideal conditions are seldom realized, and in practice the mass center lies at a distance *e* (eccentricity) from the axis of rotation, so that there is a net centrifugal force $F = me\omega^2$ acting on the rotor, where *m* is the mass of the rotor and ω is the rotational speed. This centrifugal force is balanced by reaction forces in the bearings, which tend to wear out the bearings with time.

The rotor unbalance can be divided into two types, static and dynamic. Static unbalance can be detected by placing the rotor on a pair of parallel rails. Then, the mass center will settle in the lowest position in a vertical plane through the rotation axis and below this axis. To balance the rotor statically, it is necessary to add a mass m' in the same plane at a distance *r* from the rotation axis and above this axis, where m' and *r* must be such that $m'r = me$. In this manner, the net centrifugal force on the rotor is zero. The net result of **static balancing** is to cause the mass center to coincide with the rotation axis, so that the rotor will remain in any position placed on the rails. However, unless the mass m' is placed on a line containing m and at right angles with the bearings axis, the centrifugal forces on m and m' will form a couple (Fig. 3.4.11). Static balancing is suitable when the rotor is in the form of a thin disk, in which case the couple tends to be small. Automobile tires are at times balanced statically (seems), although strictly speaking they are neither thin nor rigid.

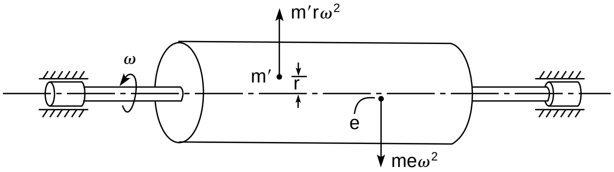


Fig. 3.4.11

In general, for practical reasons, the mass m' cannot be placed on an axis containing m and perpendicular to the bearing axis. Hence, although in static balancing the mass center lies on the rotation axis, the rotor principal axis does not coincide with the bearing axis, as shown in Fig. 3.4.12, causing the rotor to wobble during rotation. In this case, the rotor is said to be **dynamically unbalanced**. Clearly, it is highly desirable to place the mass m' so that the rotor is both statically and dynamically balanced. In this regard, note that the end planes of the rotor are convenient locations to place correcting masses. In Fig. 3.4.13, if the mass center is at a distance a from the right end, then **dynamic balance** can be achieved by placing masses $m'a/L$ and $m'(L - a)/L$ on the intersection of the plane of unbalance and the rotor left end plane and right end plane, respectively. In this manner, the resultant centrifugal force is zero and the two couples thus created are equal in value to $m'a(L - a)\omega^2/L$ and opposite in sense, so that they cancel each other. This results in a rotor completely balanced, i.e., balanced statically and dynamically.

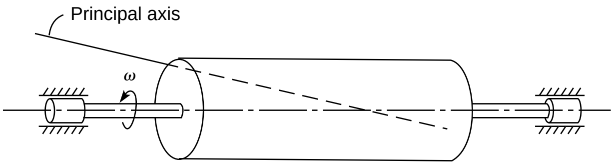


Fig. 3.4.12

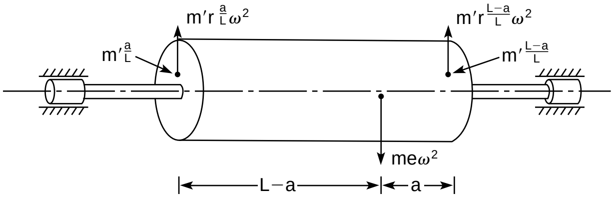


Fig. 3.4.13

The task of determining the magnitude and position of the unbalance is carried out by means of a balancing machine provided with elastically supported bearings permitting the rotor to spin (Fig. 3.4.14). The unbalance causes the bearings to oscillate laterally so that electrical pickups and stroboflash light can measure the amplitude and phase of the rotor with respect to an arbitrary rotor.

In cases in which the rotor is very long and flexible, the position of the unbalance depends on the elastic configuration of the rotor, which in turn depends on the speed of rotation, temperature, etc. In such cases, it is necessary to balance the rotor under normal operating conditions by means of a portable balancing instrument.

Inertial Unbalance of Reciprocating Engines The crank-piston mechanism of a reciprocating engine produces dynamic forces capable of causing undesirable vibrations. Rotating parts, such as the crankshaft, can be balanced. However, translating parts, such as the piston, cannot be easily balanced, and the same can be said about the connecting rod, which executes a more complex motion of combined rotation and translation.

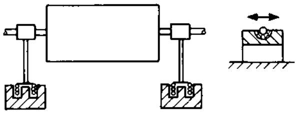


Fig. 3.4.14

In the calculation of the unbalanced forces in a single-cylinder engine, the mass of the moving parts is divided into a reciprocating mass and a rotating mass. This is done by apportioning some of the mass of the connecting rod to the piston and some to the crank end. In general, this division of the connecting rod into two lumped masses tends to cause errors in the moment of inertia, and hence in the torque equation. On the other hand, the force equation can be regarded as being accurate. (See also Sec. 8.2.)

Assuming that the rotating mass is counterbalanced, only the reciprocating mass is of concern, and the inertia force for a single-cylinder engine is

$$F = m_{\text{rec}} r\omega^2 \cos \omega t + m_{\text{rec}} \frac{r^2}{L} \omega^2 \cos 2\omega t \qquad (3.4.44)$$

where m_{rec} is the reciprocating mass, r the radius of the crank, ω the angular velocity of the crank, and L the length of the connecting rod. The first component on the right side, which alternates once per revolution, is denoted by X_1 and referred to as the **primary force**, and the second component, which is smaller and alternates twice per revolution, is denoted by X_2 and is called the **secondary force**.

In addition to the inertia force, there is an unbalanced torque about the crankshaft axis due to the reciprocating mass. However, this torque is considered together with the torque created by the power stroke, and the torsional oscillations resulting from these excitations can be mitigated by means of a pendulum-type absorber (see ''Centrifugal Pendulum Vibration Absorbers'' below) or a torsional damper.

The analysis for the single-cylinder engine can be extended to multicylinder in-line and V-block engines by superposition. For the in-line engine or one block of the V engine, the inertia force becomes

$$F = m_{\text{rec}} r\omega^2 \sum_{j=1}^{n} \cos(\omega t + \phi_j) + m_{\text{rec}} \frac{r^2}{L} \omega^2 \sum_{j=1}^{n} \cos 2(\omega t + \phi_j) \qquad (3.4.45)$$

where ϕ_j is a phase angle corresponding to the crank position associated with cylinder j and n is the number of cylinders. The vibration's force can be eliminated by proper spacing of the angular positions ϕ_j ($j = 1, 2, \ldots, n$).

Even if $F = 0$, there can be pitching and yawing moments due to the spacing of the cylinders. Table 3.4.3 gives the inertial unbalance and pitching of the primary and secondary forces for various crank-angle arrangements of n-cylinder engines.

Centrifugal Pendulum Vibration Absorbers For a rotating system, such as the crank mechanism just discussed, the exciting torques are proportional to the rotational speed ω, which varies over a wide range. Hence, for a vibration absorber to be effective, its natural frequency must be proportional to ω. The centrifugal pendulum shown in Fig. 3.4.15 is ideally suited to this task. Strictly speaking, the system of Fig. 3.4.15 represents a two-degree-of-freedom nonlinear system. However, assuming that the motion of the wheel consists of a steady rotation ω and a small harmonic oscillation at the frequency Ω, or

$$\theta(t) = \omega t + \theta_0 \sin \Omega t \qquad (3.4.46)$$

Table 3.4.3 Inertial Unbalance of Four-Stroke-per-Cycle Engines

No. n of cylinders	Crank phase angle ϕ_j	Unbalanced forces		Unbalanced pitching moments about 1st cylinder	
		Primary	Secondary	Primary	Secondary
1		X_1	X_2	—	—
2	0–180°	0	$2X_2$	ℓX_1	$2\ell X_2$
4	0–180°–180°–0	0	$4X_2$	0	$6\ell X_2$
4	0–90°–270°–180°	0	0	$\ell X_1\sqrt{1+3^2}$	0
6	0–120°–240°–240°–120°–0	0	0	0	0
8	0–180°–90°–270°–270°–90°–180°–0	0	0	0	0
90° V-8	0–90°–270°–180°	0	0	Rotating primary couple of constant magnitude $\sqrt{10}\ell X_1$ which may be completely counterbalanced	

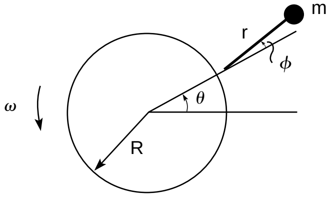


Fig. 3.4.15

and that the pendulum angle ϕ is relatively small, then the equation of motion of the pendulum reduces to the linear single-degree-of-freedom system

$$\ddot{\phi} + \omega_n^2\phi = \frac{R+r}{r}\,\Omega^2\theta_0 \sin \Omega t \qquad (3.4.47)$$

where $$\omega_n = \omega\sqrt{R/r} \qquad (3.4.48)$$

is the natural frequency of the pendulum. The torque exerted by the pendulum on the wheel is

$$T = -\frac{m(R+r)^2}{1 - r\Omega^2/R\omega^2}\,\ddot{\theta} \qquad (3.4.49)$$

so that the system behaves like a wheel with the effective mass moment of inertia

$$J_{\text{eff}} = -\frac{m(R+r)^2}{1 - r\Omega^2/R\omega^2} \qquad (3.4.50)$$

which becomes infinite when Ω is equal to the natural frequency ω_n. To suppress disturbing torques of frequency Ω several times larger than the rotational speed ω, the ratio r/R must be very small, which requires a short pendulum. The **bifilar pendulum** depicted in Fig. 3.4.16, which consists of a U-shaped counterweight that fits loosely and rolls on two pins of radius r_2 within two larger holes of equal radius r_1, represents a suitable design whereby the effective pendulum length is $r = r_1 - r_2$.

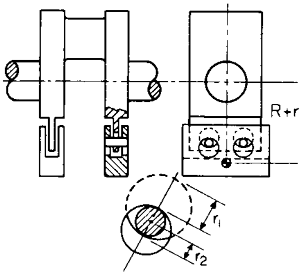


Fig. 3.4.16

Response to Periodic Excitations A problem of interest in mechanical vibrations concerns the response $x(t)$ of the cam and follower system shown in Fig. 3.4.17. As the cam rotates at a constant angular rate, the follower undergoes the periodic displacement $y(t)$, where $y(t)$ has the period T. The equation of motion is

$$m\ddot{x} + (k_1 + k_2)x = k_2 y \qquad (3.4.51)$$

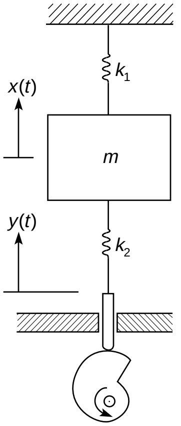


Fig. 3.4.17

Any periodic function can be expanded in a series of harmonic components in the form of the Fourier series

$$y(t) = \frac{1}{2}a_0 + \sum_{p=1}^{\infty}(a_p \cos \omega_0 t + b_p \sin p\omega_0 t) \qquad \omega_0 = 2\pi/T \qquad (3.4.52)$$

where ω_0 is called the **fundamental harmonic** and $p\omega_0$ ($p = 1, 2, \ldots$) are called **higher harmonics,** in which p is an integer. The coefficients have the expressions

$$\begin{aligned} a_p &= \frac{2}{T}\int_0^T y(t)\cos p\omega_0 t \qquad p = 0, 1, 2, \ldots \\ b_p &= \frac{2}{T}\int_0^T y(t)\sin p\omega_0 t \qquad p = 1, 2, \ldots \end{aligned} \qquad (3.4.53)$$

Note that the limits of integration can be changed, as long as the integration covers one complete period. From Eq. (3.4.27), and a companion equation for the sine counterpart, the response is

$$x(t) = \frac{k_2}{k_1 + k_2}\left[\frac{1}{2}a_0 + \sum_{p=1}^{\infty}\frac{1}{1 - (p\omega_0/\omega_n)^2} \times (a_p \cos p\omega_0 t + b_p \sin p\omega_0 t)\right] \qquad (3.4.54)$$

where $$\omega_n = \sqrt{(k_1 + k_2)/m} \qquad (3.4.55)$$

is the natural frequency of the system. Equation (3.5.54) describes a steady-state response, so that a description in terms of time is not very informative. More significant information can be extracted by plotting the amplitudes of the harmonic components versus the harmonic number. Such plots are called **frequency spectra,** and there is one for the excitation and one for the response. Equation (3.4.54) leads to the conclusion that **resonance occurs for** $p\omega_0 = \omega_n$.

As an example, consider the periodic excitation shown in Fig. 3.4.18 and use Eqs. (3.4.53) to obtain the coefficients

$$a_0 = 2A,\ a_p = 0,\ b_p = \begin{cases} 4B/p\pi & p \text{ odd} \\ 0 & p \text{ even} \end{cases} \qquad (3.4.56)$$

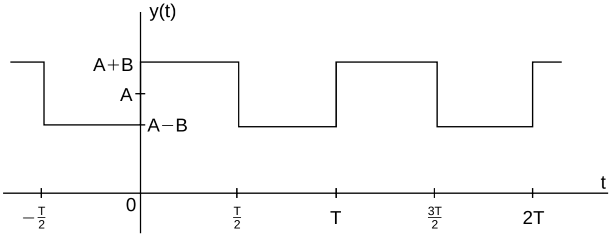


Fig. 3.4.18 Example of periodic excitation.

The excitation and response frequency spectra are displayed in Figs. 3.4.19*a* and *b*, the latter for the case in which $\omega_n = 4\omega_0$.

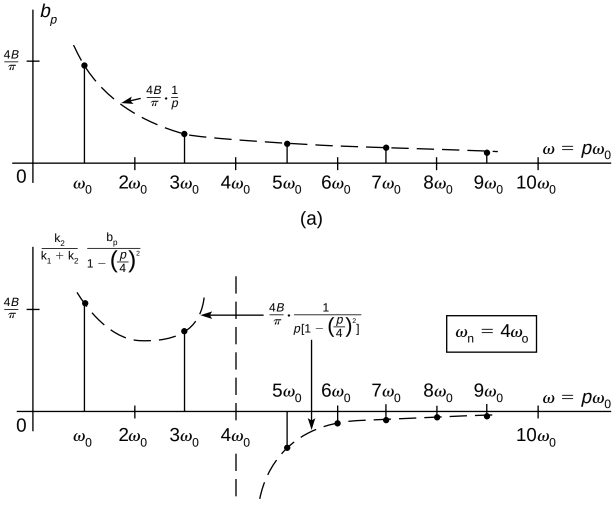


Fig. 3.4.19 (*a*) Excitation frequency spectrum; (*b*) response frequency spectrum for the periodic excitation of Fig. 3.4.18.

Unit Impulse and Impulse Response Harmonic and periodic forces represent **steady-state excitations** and **persist indefinitely**. The response to such forces is also steady state. An entirely different class of forces consists of **arbitrary,** or **transient, forces**. The term *transient* is not entirely appropriate, as some of these forces can also persist indefinitely. Concepts pivotal to the response to arbitrary forces are the unit impulse and the impulse response. The **unit impulse,** denoted by $\delta(t - a)$, represents a function of very high amplitude and defined over a very small time interval at $t = a$ such that the area enclosed is equal to 1 (Fig. 3.4.20). The **impulse response,** denoted by $g(t)$, is defined as the response of a system to a unit impulse applied at $t = 0$, with the initial conditions being equal to zero. For the mass-damper-spring system of Fig. 3.4.2, the impulse response is

$$g(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \qquad t > 0 \qquad (3.4.57)$$

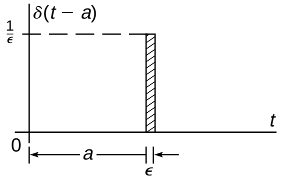


Fig. 3.4.20

Convolution Integral An arbitrary force $F(t)$ as shown in Fig. 3.4.21 can be regarded as a superposition of impulses of magnitude $F(\tau)\,d\tau$ and applied at $t = \tau$. Hence, the response to an arbitrary force can be regarded as a superposition of impulse responses $g(t - \tau)$ of magnitude $F(\tau)\,d\tau$, or

$$x(t) = \int_0^t F(\tau)g(t-\tau)\,d\tau$$
$$= \frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau)\,d\tau \qquad (3.4.58a)$$

which is called the **convolution integral** or the **superposition integral;** it can also be written in the form

$$x(t) = \int_0^t F(t-\tau)g(\tau)\,d\tau$$
$$= \frac{1}{m\,\omega_d}\int_0^t F(t-\tau)e^{-\zeta\omega_n\tau} \sin \omega_d\tau\,d\tau \qquad (3.4.58b)$$

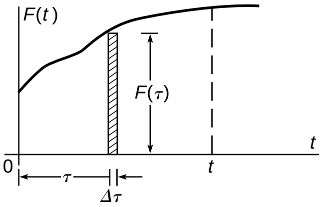


Fig. 3.4.21

Shock Spectrum Many systems are subjected on occasions to large forces applied suddenly and over periods of time that are short compared to the natural period. Such forces are capable of inflicting serious damage on a system and are referred to as **shocks**. The severity of a shock is commonly measured in terms of the maximum value of the response of a mass-spring system. The plot of the peak response versus the natural frequency is called the **shock spectrum** or **response spectrum**.

A shock $F(t)$ is characterized by its maximum value F_0, its duration T, and its shape. It is common to approximate the force by the half-sine pulse

$$F(t) = \begin{cases} F_0 \sin \omega t & \text{for } 0 < t < T = \pi/\omega \\ 0 & \text{for } t < 0 \text{ and } t > T \end{cases} \qquad (3.4.59)$$

Using the convolution integral, Eq. (3.4.58*b*) with $\zeta = 0$, the response of a mass-spring system *during the duration of the pulse* is

$$x(t) = \frac{F_0}{k[1-(\omega/\omega_n)^2]}\left(\sin \omega t - \frac{\omega}{\omega_n}\sin \omega_n t\right)$$
$$0 < t < \pi/\omega \qquad (3.4.60)$$

The maximum response is obtained when $\dot{x} = 0$ and has the value

$$x_{\max} = \frac{F_0}{k(1 - \omega/\omega_n)} \sin \frac{2i\pi}{1 + \omega_n/\omega}$$

$$i = 1, 2, \ldots ; i < \frac{1}{2}\left(1 + \frac{\omega_n}{\omega}\right) \qquad (3.4.61)$$

On the other hand, the response *after the termination of the pulse* is

$$x(t) = \frac{F_0\omega_n^2/\omega}{k[1 - (\omega_n/\omega)^2]}[\cos \omega_n t + \cos \omega_n(t - T)] \qquad (3.4.62)$$

which has the maximum value

$$x_{\max} = \frac{2\,F_0\,\omega_n/\omega}{k[1 - (\omega_n/\omega)^2]} \cos \frac{\pi\omega_n}{2\omega} \qquad (3.4.63)$$

The shock spectrum is the plot $x_{\max}$ versus ω_n/ω. For $\omega_n < \omega$, the maximum response is given by Eq. (3.4.63) and for $\omega_n > \omega$ by Eq. (3.4.61). The shock spectrum is shown in Fig. 3.4.22 in the form of the nondimensional plot $x_{\max}k/F_0$ versus ω_n/ω.

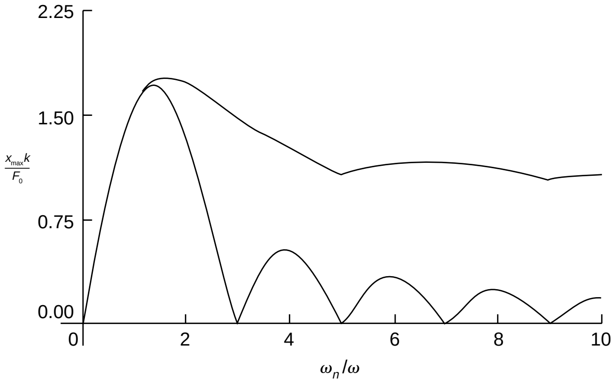


Fig. 3.4.22

MULTI-DEGREE-OF-FREEDOM SYSTEMS

Equations of Motion Many vibrating systems require more elaborate models than a single-degree-of-freedom system, such as the **multi-degree-of-freedom system** shown in Fig. 3.4.23. By using Newton's second law for each of the n masses m_i ($i = 1, 2, \ldots, n$), the equations of motion can be written in the form

$$m_i\ddot{x}_i(t) + \sum_{j=1}^{n} c_{ij}\dot{x}_j(t) + \sum_{j=1}^{n} k_{ij}x_j(t) = F_i(t)$$

$$i = 1, 2, \ldots, n \qquad (3.4.64)$$

where $x_i(t)$ is the displacement of mass m_i, $F_i(t)$ is the force acting on m_i, and c_{ij} and k_{ij} are **damping** and **stiffness coefficients,** respectively. The matrix form of Eqs. (3.4.64) is

$$M\ddot{\mathbf{x}}(t) + C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{F}(t) \qquad (3.4.65)$$

in which $\mathbf{x}(t)$ is the n-dimensional displacement vector, $\mathbf{F}(t)$ the corresponding force vector, M the **mass matrix,** C the **damping matrix,** and K the **stiffness matrix,** all three symmetric matrices. (In the present case the mass matrix is diagonal, but in general it is not, although it is symmetric.)

Response of Undamped Systems to Harmonic Excitations Let the harmonic excitation have the form

$$\mathbf{F}(t) = \mathbf{F}_0 \sin \omega t \qquad (3.4.66)$$

where $\mathbf{F}_0$ is a constant vector and ω is the excitation, or driving frequency. The response to the harmonic excitation is a **steady-state** response and can be expressed as

$$\mathbf{x}(t) = Z^{-1}(\omega)\mathbf{F}_0 \sin \omega t \qquad (3.4.67)$$

where $Z^{-1}(\omega)$ is the inverse of the **impedance matrix** $Z(\omega)$. In the absence of damping, the impedance matrix is

$$Z(\omega) = K - \omega^2 M \qquad (3.4.68)$$

Undamped Vibration Absorbers When a mass-spring system m_1, k_1 is subjected to a harmonic force with the frequency equal to the natural frequency, resonance occurs. In this case, it is possible to add a second mass-spring system m_2,k_2 so designed as to produce a two-degree-of-freedom system with the response of m_1 equal to zero. We refer to m_1, k_1 as the **main system** and to m_2,k_2 as the **vibration absorber**. The resulting two-degree-of-freedom system is shown in Fig. 3.4.24 and has the impedance matrix

$$Z(\omega) = \begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \qquad (3.4.69)$$

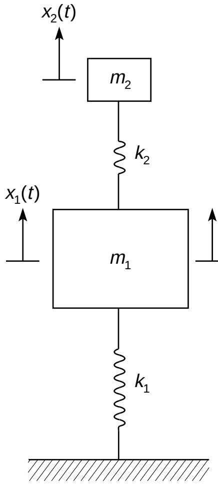


Fig. 3.4.24

Inserting Eq. (3.4.69) into Eq. (3.4.67), together with $F_1(t) = F_1 \sin \omega t$, $F_2(t) = 0$, write the steady-state response in the form

$$x_1(t) = X_1(\omega) \sin \omega t \qquad (3.4.70a)$$

$$x_2(t) = X_2(\omega) \sin \omega t \qquad (3.4.70b)$$

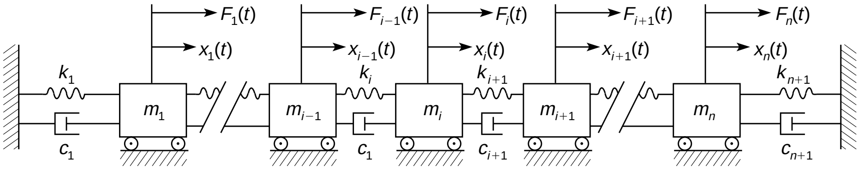


Fig. 3.4.23

where the amplitudes are given by

$$X_1(\omega) = \frac{[1 - (\omega/\omega_a)^2]x_{st}}{[1 + \mu(\omega_a/\omega_n)^2 - (\omega/\omega_n)^2][1 - (\omega/\omega_a)^2] - \mu(\omega_a/\omega_n)^2} \qquad (3.4.71a)$$

$$X_2(\omega) = \frac{x_{st}}{[1 + \mu(\omega_a/\omega_n)^2 - (\omega/\omega_n)^2][1 - (\omega/\omega_a)^2] - \mu(\omega_a/\omega_n)^2} \qquad (3.4.71b)$$

in which

$\omega_n = \sqrt{k_1/m_1}$ = the natural frequency of the main system alone

$\omega_a = \sqrt{k_2/m_2}$ = the natural frequency of the absorber alone

$x_{st} = F_1/k_1$ = the static deflection of the main system

$\mu = m_2/m_1$ = the ratio of the absorber mass to the main mass

From Eqs. (3.4.70*a*) and (3.4.71*a*), we conclude that if we choose m_2 and k_2 such that $\omega_a = \omega$, the response $x_1(t)$ of the main mass is zero. Moreover, from Eqs. (3.4.70*b*) and (3.4.71*b*),

$$x_2(t) = -\left(\frac{\omega_n}{\omega_a}\right)^2 \frac{x_{st}}{\mu} \sin \omega t = -\frac{F_1}{k_2} \sin \omega t \qquad (3.4.72)$$

so that the force in the absorber spring is

$$k_2 x_2(t) = -F_1 \sin \omega t \qquad (3.4.73)$$

Hence, *the absorber exerts a force on the main mass balancing exactly the applied force F_1 sin ωt.*

A vibration absorber designed for a given operating frequency ω can perform satisfactorily for operating frequencies that vary slightly from ω. In this case, the motion of m_1 is not zero, but its amplitude tends to be very small, as can be verified from a frequency response plot $X_1(\omega)/x_{st}$ versus ω/ω_n; Fig. 3.4.25 shows such a plot for $\mu = 0.2$ and $\omega_n = \omega_a$. The shaded area indicates the range in which the performance can be regarded as satisfactory. Note that the thin line in Fig. 3.4.25 represents the frequency response of the main system alone. Also note that the system resulting from the combination of the main system and the absorber has two resonance frequencies, but they are removed from the operating frequency $\omega = \omega_n = \omega_a$.

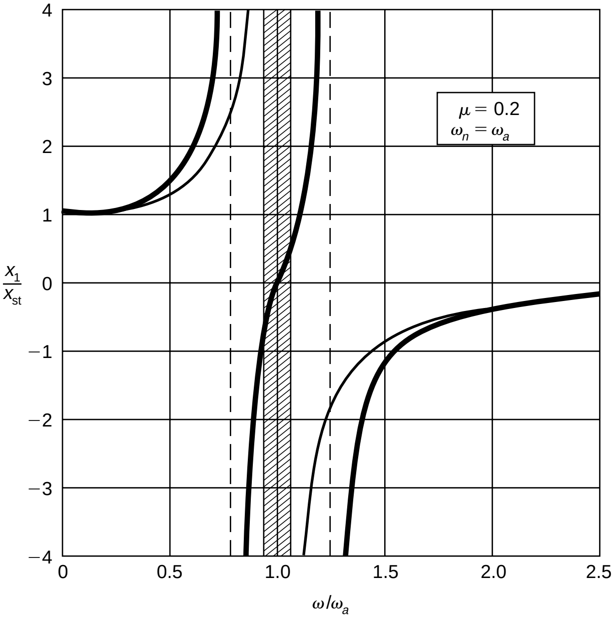


Fig. 3.4.25

Natural Modes of Vibration In the absence of damping and external forces, Eq. (3.4.65) reduces to the free-vibration equation

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0} \qquad (3.4.74)$$

which has the harmonic solution

$$\mathbf{x}(t) = \mathbf{u} \cos(\omega t - \phi) \qquad (3.4.75)$$

where **u** is a constant vector, ω a frequency of oscillation, and ϕ a phase angle. Introduction of Eq. (3.4.75) into Eq. (3.4.74) and division through by $\cos(\omega t - \phi)$ results in

$$K\mathbf{u} = \omega^2 M\mathbf{u} \qquad (3.4.76)$$

which represents a set of n simultaneous algebraic equations known as the **eigenvalue problem**. It has n solutions consisting of the **eigenvalues** ω_r^2; the square roots represent the **natural frequencies** ω_r ($r = 1, 2, \ldots, n$). Moreover, to each natural frequency ω_r there corresponds a vector $\mathbf{u}_r$ ($r = 1, 2, \ldots, n$) called **eigenvector,** or **modal vector,** or **natural mode**. The modal vectors possess the **orthogonality property,** or

$$\mathbf{u}_s^T M\mathbf{u}_r = 0 \qquad (3.4.77a)$$
$$\mathbf{u}_s^T K\mathbf{u}_r = 0 \qquad (3.4.77b)$$

(for $r, s = 1, 2, \ldots, n$; $r \neq s$), in which $\mathbf{u}_s^T$ is the **transpose** of $\mathbf{u}_s$, a row vector. It is convenient to adjust the magnitude of the modal vectors so as to satisfy

$$\mathbf{u}_r^T M\mathbf{u}_r = 1 \qquad (3.4.78a)$$
$$\mathbf{u}_r^T K\mathbf{u}_r = \omega_r^2 \qquad (3.4.78b)$$

(for $r = 1, 2, \ldots, n$), a process known as normalization, in which case $\mathbf{u}_r$ are called **normal modes**. Note that the normalization process involves Eq. (3.4.78*a*) alone, as Eq. (3.4.78*b*) follows automatically. The solution of the eigenvalue problem can be obtained by a large variety of computational algorithms (Meirovitch, ‘‘Principles and Techniques of Vibrations,’’ Prentice-Hall). Commercially, they are available in software packages for numerical computations, such as MATLAB.

The actual solution of Eq. (3.4.74) is obtained below in the context of the transient response.

Transient Response of Undamped Systems From Eq. (3.4.65), the vibration of undamped systems satisfies the equation

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{F}(t) \qquad (3.4.79)$$

where $\mathbf{F}(t)$ is an arbitrary force vector. In addition, the displacement and velocity vectors must satisfy the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(0) = \mathbf{v}_0$. The solution of Eq. (3.4.79) has the form

$$\mathbf{x}(t) = \sum_{r=1}^{n} \mathbf{u}_r q_r(t) \qquad (3.4.80)$$

in which $\mathbf{u}_r$ are the modal vectors and $\mathbf{q}_r(t)$ are associated **modal coordinates**. Inserting Eq. (3.4.80) into Eq. (3.4.79), premultiplying the result by $\mathbf{u}_s^T$, and using Eqs. (3.4.77) and (3.4.78) we obtain the modal equations

$$\ddot{q}_r(t) + \omega_r^2 q_r(t) = Q_r(t) \qquad r = 1, 2, \ldots, n \qquad (3.4.81)$$

where

$$Q_r(t) = \mathbf{u}_r^T \mathbf{F}(t) \qquad r = 1, 2, \ldots, n \qquad (3.4.82)$$

are **modal forces**. Equations (3.4.81) resemble the equation of single-degree-of-freedom system and have the solution

$$q_r(t) = \frac{1}{\omega_r} \int_0^t Q_r(t - \tau) \sin \omega_r \tau \, d\tau + q_r(0) \cos \omega_r t + \frac{\dot{q}_r(0)}{\omega_r} \sin \omega_r t \qquad r = 1, 2, \ldots, n \qquad (3.4.83)$$

where

$$q_r(0) = \mathbf{u}_r^T M\mathbf{x}_0 \qquad (3.4.84a)$$
$$\dot{q}_r(0) = \mathbf{u}_r^T M\mathbf{v}_0 \qquad (3.4.84b)$$

(for $r = 1, 2, \ldots, n$) are **initial modal displacements and velocities,** respectively. The solution to both external forces and initial excitations is obtained by inserting Eqs. (3.4.83) into Eq. (3.4.80).

Systems with Proportional Damping When the system is damped, the response does not in general have the form of Eq. (3.4.80), and a more involved approach is necessary (Meirovitch, ''Elements of Vibration Analysis,'' 2d ed., McGraw-Hill). In the special case in which the damping matrix C is proportional to the mass matrix M or the stiffness matrix K, or is a linear combination of M and K, the preceding approach yields the modal equations

$$\ddot{q}_r(t) + 2\zeta_r\omega_r\dot{q}_r(t) + \omega_r^2 q_r(t) = Q_r(t) \qquad r = 1, 2, \ldots, n \qquad (3.4.85)$$

where ζ_r are **modal damping factors**. Equations (3.4.85) have the solution

$$q_r(t) = \frac{1}{\omega_{dr}}\int_0^t Q_r(t-\tau)e^{-\zeta_r\omega_r\tau}\sin\omega_{dr}\tau\, d\tau + \frac{q_r(0)}{\sqrt{1-\zeta_r^2}}e^{-\zeta_r\omega_r t}\cos(\omega_{dr}t - \psi_r) + \frac{\dot{q}_r(0)}{\omega_{dr}}\sin\omega_{dr}t \qquad r = 1, 2, \ldots, n \qquad (3.4.86)$$

in which

$$\omega_{dr} = \omega_r\sqrt{1-\zeta_r^2} \qquad r = 1, 2, \ldots, n \qquad (3.4.87)$$

is the damped frequency in the rth mode and

$$\psi_r = \tan^{-1}\frac{\zeta_r}{\sqrt{1-\zeta_r^2}} \qquad r = 1, 2, \ldots, n \qquad (3.4.88)$$

is a phase angle associated with the rth mode. The quantities $Q_r(t)$, $q_r(0)$, and $\dot{q}_r(0)$ remain as defined by Eqs. (3.4.82), (3.4.84*a*), and (3.4.84*b*), respectively.

DISTRIBUTED-PARAMETER SYSTEMS

Vibration of Rods, Shafts, and Strings The axial vibration of rods is described by the equation

$$-\frac{\partial}{\partial x}\left\{EA(x)\frac{\partial u(x,t)}{\partial x}\right\} + m(x)\frac{\partial^2 u(x,t)}{\partial t^2} = f(x,t) \qquad 0 < x < L \qquad (3.4.89)$$

where $u(x, t)$ is the axial displacement, $f(x, t)$ the axial force per unit length, E the modulus of elasticity, $A(x)$ the cross-sectional area, and $m(x)$ the mass per unit length. The solution $u(x, t)$ is subject to one boundary condition at each end.

Before attempting to solve Eq. (3.4.89), consider the free vibration problem, $f(x, t) = 0$. The solution of the latter problem is harmonic and can be expressed as

$$u(x,t) = U(x)\cos(\omega t - \phi) \qquad (3.4.90)$$

where $U(x)$ is the amplitude, ω the frequency, and ϕ an inconsequential phase angle. Inserting Eq. (3.4.90) into Eq. (3.4.89) with $f(x, t) = 0$ and dividing through by $\cos(\omega t - \phi)$, we conclude that $U(x)$ and ω must satisfy the *eigenvalue problem*

$$-\frac{d}{dx}\left\{EA(x)\frac{dU(x)}{dx}\right\} = \omega^2 m(x)U(x) \qquad 0 < x < L \qquad (3.4.91)$$

where $U(x)$ must satisfy one boundary condition at each end. At a fixed end the displacement U must be zero and at a free end the axial force $EA\, dU/dx$ is zero.

Exact solutions of the eigenvalue problem are possible in only a few cases, mostly for uniform rods, in which case Eq. (3.4.91) reduces to

$$\frac{d^2U(x)}{dx^2} + \beta^2 U(x) = 0 \qquad \beta^2 = \frac{\omega^2 m}{EA} \qquad 0 < x < L \qquad (3.4.92)$$

whose solution is

$$U(x) = A\sin\beta x + B\cos\beta x \qquad (3.4.93)$$

where A and B are constants of integration, determined from specified boundary conditions. In the case of a **fixed-free** rod, the boundary conditions are

$$U(0) = 0 \qquad (3.4.94a)$$

$$EA\left.\frac{dU}{dx}\right|_{x=L} = 0 \qquad (3.4.94b)$$

Condition (3.4.94*a*) gives $B = 0$ and condition (3.4.94*b*) yields the **characteristic equation**

$$\cos\beta L = 0 \qquad (3.4.95)$$

which has the infinity of solutions

$$\beta_r L = \frac{(2r-1)\pi}{2} \qquad r = 1, 2, \ldots \qquad (3.4.96)$$

where β_r represent the *eigenvalues;* they are related to the *natural frequencies* ω_r by

$$\omega_r = \beta_r\sqrt{\frac{EA}{m}} = \frac{(2r-1)\pi}{2}\sqrt{\frac{EA}{mL^2}} \qquad r = 1, 2, \ldots \qquad (3.4.97)$$

From Eq. (3.4.93), the **normal modes** are

$$U_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{(2r-1)\pi x}{2L} \qquad r = 1, 2, \ldots \qquad (3.4.98)$$

For a **fixed-fixed** rod, the natural frequencies and normal modes are

$$\omega_r = r\pi\sqrt{\frac{EA}{mL^2}} \qquad U_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{r\pi x}{L} \qquad r = 1, 2, \ldots \qquad (3.4.99)$$

and for a **free-free** rod they are

$$\omega_0 = 0 \qquad U_0 = \sqrt{\frac{1}{mL}} \qquad (3.4.100a)$$

$$\omega_r = r\pi\sqrt{\frac{EA}{mL^2}} \qquad U_r(x) = \sqrt{\frac{2}{mL}}\cos\frac{r\pi x}{L} \qquad r = 1, 2, \ldots \qquad (3.4.100b)$$

Note that U_0 represents a **rigid-body mode,** with zero natural frequency.

In every case the modes are orthogonal, satisfying the conditions

$$\int_0^L mU_s(x)U_r(x)\, dx = 0 \qquad (3.4.101a)$$

$$-\int_0^L U_s(x)\frac{d}{dx}\left[EA\frac{dU_r(x)}{dx}\right]dx = 0 \qquad (3.4.101b)$$

(for $r, s = 0, 1, 2, \ldots, r \neq s$) and have been normalized to satisfy the relations

$$\int_0^L mU_r^2(x)\, dx = 1 \qquad (3.4.102a)$$

$$-\int_0^L U_r(x)\frac{d}{dx}\left[EA\frac{dU_r(x)}{dx}\right]dx = \omega_r^2 \qquad (3.4.102b)$$

(for $r = 0, 1, 2, \ldots$). Note that the orthogonality of the normal modes extends to the rigid-body mode.

The response of the rod has the form

$$u(x,t) = \sum_{r=1}^{\infty} U_r(x)q_r(t) \qquad (3.4.103)$$

Introducing Eq. (3.4.103) into Eq. (3.4.89), multiplying through by $U_s(x)$, integrating over the length of the rod, and using Eqs. (3.4.101) and (3.4.102) we obtain the **modal equations**

$$\ddot{q}_r(t) + \omega_r^2 q_r(t) = Q_r(t) \qquad r = 1, 2, \ldots \qquad (3.4.104)$$

where

$$Q_r(t) = \int_0^L U_r(x)f(x,t)\, dx \qquad r = 1, 2, \ldots \qquad (3.4.105)$$

Table 3.4.4 Analogous Quantities for Rods, Shafts, and Strings

	Rods	Shafts	Strings
Displacement	Axial—$u(x, t)$	Torsional—$\theta(x, t)$	Transverse—$w(x, t)$
Inertia (per unit length)	Mass—$m(x)$	Mass polar moment of inertia—$I(x)$	Mass—$\rho(x)$
Stiffness	Axial—$EA(x)$ E = Young's modulus $A(x)$ = cross-sectional area	Torsional—$GJ(x)$ G = shear modulus $J(x)$ = area polar moment of inertia	Tension—$T(x)$
Load (per unit length)	Force—$f(x, t)$	Moment—$m(x, t)$	Force—$f(x, t)$

are the **modal forces.** Equations (3.4.104) resemble Eqs. (3.4.81) entirely; their solution is given by Eqs. (3.4.83). The displacement of the rod is obtained by inserting Eqs. (3.4.83) into Eq. (3.4.103).

As an example, consider the response of a uniform fixed-free rod to the uniformly distributed impulsive force

$$f(x, t) = \hat{f}_0\delta(t) \qquad (3.4.106)$$

Inserting Eqs. (3.4.98) and (3.4.106) into Eq. (3.4.105), we obtain the modal forces

$$\begin{aligned} Q_r(t) &= \sqrt{\frac{2}{mL}}\int_0^L \sin\frac{(2r-1)\pi x}{2L}\,\hat{f}_0\delta(t)\,dx \\ &= \frac{2}{(2r-1)\pi}\sqrt{\frac{2L}{m}}\,\hat{f}_0\delta(t) \qquad r = 1, 2, \ldots \qquad (3.4.107) \end{aligned}$$

so that, from Eqs. (3.4.83), the modal displacements are

$$\begin{aligned} q_r(t) &= \frac{1}{\omega_r}\,\frac{2}{(2r-1)\pi}\sqrt{\frac{2L}{m}}\,\hat{f}_0\int_0^t \delta(t-\tau)\sin\omega_r\tau d\tau \\ &= \left[\frac{2}{(2r-1)\pi}\right]^2\sqrt{\frac{2L^3}{EA}}\,\hat{f}_0 \sin\frac{(2r-1)\pi}{2}\sqrt{\frac{EA}{mL^2}}\,t \\ & \qquad r = 1, 2, \ldots \qquad (3.4.108) \end{aligned}$$

Finally, from Eq. (3.4.103), the response is

$$\begin{aligned} u(x, t) &= \frac{8\hat{f}_0 L}{\pi^2}\sqrt{\frac{1}{mEA}}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}\sin\frac{(2r-1)\pi x}{2L} \\ & \qquad \times \sin\frac{(2r-1)\pi}{2}\sqrt{\frac{EA}{mL^2}}\,t \qquad (3.4.109) \end{aligned}$$

The torsional vibration of shafts and the transverse vibration of strings are described by the same differential equation and boundary conditions as the axial vibration of rods, except that the nature of the displacement, inertia and stiffness parameters, and external excitations differs, as indicated in Table 3.4.4.

Bending Vibration of Beams The procedure for evaluating the response of beams in transverse vibration is similar to that for rods, the main difference arising in the stiffness term. The differential equation for beams in bending is

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] + m(x)\,\frac{\partial^2 w(x, t)}{\partial t^2} \\ = f(x, t) \qquad 0 < x < L \qquad (3.4.110) \end{aligned}$$

in which $w(x, t)$ is the transverse displacement, $f(x, t)$ the force per unit length, $I(x)$ the cross-sectional area moment of inertia, and $m(x)$ the mass per unit length. The solution $w(x, t)$ must satisfy two boundary conditions at each end.

The eigenvalue problem is described by the differential equation

$$\frac{d^2}{dx^2}\left[EI(x)\,\frac{d^2W(x)}{dx^2}\right] = \omega^2 m(x)W(x) \qquad 0 < x < L \qquad (3.4.111)$$

and two boundary conditions at each end, depending on the type of support. Some possible boundary conditions are given in Table 3.4.5. The solution of the eigenvalue problem consists of the natural frequencies ω_r and natural modes $W_r(x)$ ($r = 1, 2, \ldots$). The first five normalized natural frequencies of uniform beams with six different boundary conditions are listed in Table 3.4.6. The normal modes for the hinged-hinged beam are

$$W_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{r\pi x}{L} \qquad r = 1, 2, \ldots \qquad (3.4.112)$$

The normal modes for the remaining beam types are more involved and they involve both trigonometric and hyperbolic functions (Meirovitch, ''Elements of Vibration Analysis,'' 2d ed.) The modes for every beam type are orthogonal and can be used to obtain the response $w(x, t)$ in the form of a series similar to Eq. (3.4.103).

Table 3.4.5 Quantities Equal to Zero at Boundary

Boundary type	Displacement W	Slope dW/dx	Bending moment EId^2W/dx^2	Shearing force $d(EId^2W/dx^2)/dx$
Hinged	✓		✓	
Clamped	✓	✓		
Free			✓	✓

Vibration of Membranes A **membrane** is a very thin sheet of material stretched over a two-dimensional domain enclosed by one or two nonintersecting boundaries. It can be regarded as the two-dimensional counterpart of the string. Like a string, it derives the ability to resist transverse displacements from tension, which acts in all directions in the plane of the membrane and at all its points. It is commonly assumed that the tension is uniform and does not change as the membrane de-

Table 3.4.6 Normalized Natural Frequencies for Various Beams

Beam type	$\omega_1\sqrt{mL^4/EI}$	$\omega_2\sqrt{mL^4/EI}$	$\omega_3\sqrt{mL^4/EI}$	$\omega_4\sqrt{mL^4/EI}$	$\omega_5\sqrt{mL^2/EI}$
Hinged–hinged	π^2	$4\pi^2$	$9\pi^2$	$16\pi^2$	$25\pi^2$
Clamped–free	1.875^2	4.694^2	7.855^2	10.996^2	14.137^2
Free–free	0	0	$(1.506\pi)^2$	$(2.500\pi)^2$	$(3.500\pi)^2$
Clamped–clamped	$(1.506\pi)^2$	$(2.500\pi)^2$	$(3.500\pi)^2$	$(4.500\pi)^2$	$(5.500\pi)^2$
Clamped–hinged	3.927^2	7.069^2	10.210^2	13.352^2	16.493^2
Hinged–free	0	3.927^2	7.069^2	10.210^2	13.352^2

flects. The general procedure for calculating the response of membranes remains the same as for rods and beams, but there is one significant new factor, namely, the shape of the boundary, which dictates the type of coordinates to be used. For rectangular membranes cartesian coordinates must be used, and for circular membranes polar coordinates are indicated.

The differential equation for the transverse vibration of membranes is

$$-T\nabla^2 w + \rho \frac{\partial^2 w}{\partial t^2} = f \qquad (3.4.113)$$

which must be satisfied at every interior point of the membrane, where w is the transverse displacement, f the transverse force per unit area, T the tension, and ρ the mass per unit area. Moreover, ∇^2 is the Laplacian operator, whose expression depends on the coordinates used. The solution w must satisfy one boundary condition at every boundary point. Using the established procedure, the eigenvalue problem is described by the differential equation

$$-T\nabla^2 W = \omega^2 \rho W \qquad (3.4.114)$$

where W is the displacement amplitude; it must satisfy one boundary condition at every point of the boundary.

Consider a **rectangular membrane fixed** at $x = 0, a$ and $y = 0, b$, in which case the Laplacian operator in terms of the cartesian coordinates x and y has the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad (3.4.115)$$

The boundary conditions are $W(0, y) = W(a, y) = W(x, 0) = W(x, b) = 0$. The natural frequencies are

$$\omega_{mn} = \pi \sqrt{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \frac{T}{\rho}} \qquad m, n = 1, 2, \ldots \qquad (3.4.116)$$

and the normal modes are

$$W_{mn}(x, y) = \frac{2}{\sqrt{\rho ab}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad m, n = 1, 2, \ldots \qquad (3.4.117)$$

The modes satisfy the orthogonality conditions

$$\int_0^a \int_0^b \rho W_{mn}(x, y) W_{rs}(x, y)\, dx\, dy = 0, \qquad m \neq r \text{ and/or } n \neq s \qquad (3.4.118a)$$

$$-\int_0^a \int_0^b W_{mn}(x, y) T \nabla^2 W_{rs}(x, y)\, dx\, dy = 0, \qquad m \neq r \text{ and/or } n \neq s \qquad (3.4.118b)$$

and have been normalized so that $\int_0^a \int_0^b \rho W_{mn}^2(x, y)\, dx\, dy = 1 (m, n = 1, 2, \ldots)$. Note that, because the problem is two-dimensional, it is necessary to identify the natural frequencies and modes by two subscripts. With this exception, the procedure for obtaining the response is the same as for rods and beams.

Next, consider a **uniform circular membrane fixed at** $r = a$. In this case, the Laplacian operator in terms of the polar coordinates r and θ is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad (3.4.119)$$

The natural modes for circular membranes are appreciably more involved than for rectangular membranes. They are products of Bessel functions of $\omega_{mn} r$ and trigonometric functions of $m\theta$, where $m = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$. The modes are given in Meirovitch, ‘‘Principles and Techniques of Vibrations,’’ Prentice-Hall. Table 3.4.7 gives the normalized natural frequencies $\omega_{mn}^* = (\omega_{mn}/2\pi)\sqrt{\rho a^2/T}$ corresponding to $m = 0, 1, 2$ and $n = 1, 2, 3$. The modes satisfy the orthogonality relations

$$\int_0^a \int_0^{2\pi} \rho W_{mn}(r, \theta) W_{rs}(r, \theta) r\, dr\, d\theta = 0 \qquad m \neq r \text{ and/or } n \neq s \qquad (3.4.120a)$$

$$-\int_0^a \int_0^{2\pi} W_{mn}(r, \theta) T \nabla^2 W_{rs}(r, \theta) r\, dr\, d\theta = 0 \qquad m \neq r \text{ and/or } n \neq s \qquad (3.4.120b)$$

Table 3.4.7 Circular Membrane Normalized Natural Frequencies
$\omega_{mn}^* = (\omega_{mn}/2\pi)\sqrt{\rho a^2/T}$

	n		
m	1	2	3
0	0.3827	0.8786	1.3773
1	0.6099	1.1165	1.6192
2	0.8174	1.3397	1.8494

The response of circular membranes is obtained in the usual manner.

Bending Vibration of Plates Consider **plates** whose behavior is governed by the elementary plate theory, which is based on the following assumptions: (1) deflections are small compared to the plate thickness; (2) the normal stresses in the direction transverse to the plate are negligible; (3) there is no force resultant on the cross-sectional area of a plate differential element: the middle plane of the plate does not undergo deformations and represents a neutral plane, and (4) any straight line normal to the middle plane remains so during bending. Under these assumptions, the differential equation for the bending vibration of plates is

$$D\nabla^4 w + m \frac{\partial^2 w}{\partial t^2} = f \qquad (3.4.121)$$

and is to be satisfied at every interior point of the plate, where w is the transverse displacement, f the transverse force per unit area, m the mass per unit area, $D = Eh^3/12(1 - v^2)$ the plate flexural rigidity, E Young’s modulus, h the plate thickness, and v Poisson’s ratio. Moreover, ∇^4 is the biharmonic operator. The solution w must satisfy two boundary conditions at every point of the boundary. The eigenvalue problem is defined by the differential equation

$$D\nabla^4 W = \omega^2 m W \qquad (3.4.122)$$

and corresponding boundary conditions.

Consider a **rectangular plate simply supported** at $x = 0, a$ and $y = 0, b$. Because of the shape of the plate, we must use cartesian coordinates, in which case the biharmonic operator has the expression

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \qquad (3.4.123)$$

Moreover, the boundary conditions are $W = 0$ and $\partial^2 W/\partial x^2 = 0$ for $x = 0, a$ and $W = 0$ and $\partial^2 W/\partial y^2 = 0$ for $y = 0, b$. The natural frequencies are

$$\omega_{mn} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \sqrt{\frac{D}{m}} \qquad m, n = 1, 2, \ldots \qquad (3.4.124)$$

and no confusion should arise because the same symbol is used for one of the subscripts and for the mass per unit area. The corresponding normal modes are

$$W_{mn}(x, y) = \frac{2}{\sqrt{mab}} \sin \frac{m\pi x}{n} \sin \frac{n\pi y}{b} \qquad m, n = 1, 2, \ldots \qquad (3.4.125)$$

and they are recognized as being the same as for rectangular membranes fixed at all boundaries.

A **circular plate** requires use of polar coordinates, so that the biharmonic operator has the form

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \qquad (3.4.126)$$

Table 3.4.8 Circular Plate Normalized Natural Frequencies
$\omega_{mn}^* = (\omega_{mn}(a/\pi)^2\sqrt{m/D}$

	n		
m	1	2	3
0	1.015^2	2.007^2	3.000^2
1	1.468^2	2.483^2	3.490^2
2	1.879^2	2.992^2	4.000^2

Consider a **plate clamped** at $r = a$, in which case the boundary conditions are $W(r, \theta) = 0$ and $\partial W(r, \theta)/\partial r = 0$ at $r = a$. In addition, the solution must be finite at every interior point in the plate, and in particular at $r = 0$. The natural modes have involved expressions; they are given in Meirovitch, ''Principles and Techniques of Vibrations,'' Prentice-Hall. Table 3.4.8 lists the normalized natural frequencies $\omega_{mn}^* = \omega_{mn}(a/\pi)^2\sqrt{m/D}$ corresponding to $m = 0, 1, 2$ and $n = 1, 2, 3$.

The natural modes of the plates are orthogonal and can be used to obtain the response to both initial and external excitations.

APPROXIMATE METHODS FOR DISTRIBUTED SYSTEMS

Rayleigh's Energy Method The eigenvalue problem contains vital information concerning vibrating systems, namely, the natural frequencies and modes. In the majority of practical cases, exact solutions to the eigenvalue problem for distributed systems are not possible, so that the interest lies in **approximate solutions**. This is often the case when the mass and stiffness are distributed nonuniformly and/or the boundary conditions cannot be satisfied, the latter in particular for two-dimensional systems with irregularly shaped boundaries.

When the objective is to estimate the lowest natural frequency, Rayleigh's energy method has few equals. As discussed earlier, free vibration of undamped systems is harmonic and can be expressed as

$$w(x, t) = W(x)\cos(\omega t - \phi) \tag{3.4.127}$$

where $W(x)$ is the displacement amplitude, ω the free vibration frequency, and ϕ an inconsequential phase angle. The kinetic energy represents an integral involving the velocity squared. Hence, using Eq. (3.4.127), the kinetic energy can be written in the form

$$T(t) = \frac{1}{2}\int_0^L m(x)\left[\frac{\partial w(x, t)}{\partial t}\right]^2 dx = \omega^2 T_{\text{ref}}\sin^2(\omega t - \phi) \tag{3.4.128}$$

where

$$T_{\text{ref}} = \frac{1}{2}\int_0^L m(x)W^2(x)\,dx \tag{3.4.129}$$

is called the **reference kinetic energy**. The form of the potential energy is system-dependent, but in general is an integral involving the square of the displacement and of its derivatives with respect to the spatial coordinates (see Table 3.4.9). It can be expressed as

$$V(t) = V_{\max}\cos^2(\omega t - \phi) \tag{3.4.130}$$

where $V_{\max}$ is the maximum potential energy, which can be obtained by simply replacing $w(x, t)$ by $W(x)$ in $V(t)$. Using the principle of conservation of energy in conjunction with Eqs. (3.4.128) and (3.4.130), we can write

$$E = T + V = T_{\max} + 0 = 0 + V_{\max} \tag{3.4.131}$$

in which

$$T_{\max} = \omega^2 T_{\text{ref}} \tag{3.4.132}$$

It follows that

$$\omega^2 = \frac{V_{\max}}{T_{\text{ref}}} \tag{3.4.133}$$

Equation (3.4.133) represents **Rayleigh's quotient,** which has the remarkable property that it has a minimum value for $W(x) = W_1(x)$, the minimum value being ω_1^2. Rayleigh's energy method amounts to selecting a trial function $W(x)$ reasonably close to the lowest natural mode $W_1(x)$, inserting this function into Rayleigh's quotient, and carrying out the indicated integrations. Then, ω^2 will be one order of magnitude closer to the lowest eigenvalue ω_1^2 than $W(x)$ is to $W_1(x)$, thus providing a good estimate ω of the lowest natural frequency ω_1. Quite often, the static deformation of the system acted on by loads proportional to the mass distribution is a good choice. In some cases, the lowest mode of a related simpler system can yield good results.

As an example, estimate the lowest natural frequency of a uniform bar in axial vibration with a mass M attached at $x = L$ (Fig. 3.4.26) for the three trial functions (1) $U(x) = x/L$; (2) $U(x) = (1 + M/mL)(x/L) - (x/L)^2/2$, representing the static deformation; and (3) $U(x) = \sin \pi x/2L$, representing the lowest mode of the bar without the mass M. The Rayleigh quotient for this bar is

$$\omega^2 = \frac{\int_0^L EA(x)[dU(x)/dx]^2\,dx}{\int_0^L m(x)U^2(x)\,dx + MU^2(L)} \tag{3.4.134}$$

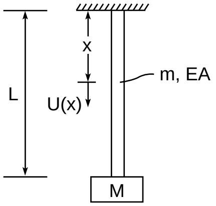


Fig. 3.4.26

The results are:

1. $$\omega^2 = \frac{\int_0^L EA(1/L)^2\,dx}{\int_0^L m(x/L)^2\,dx + M} = \frac{EA}{(M + mL/3)L}$$

2. $$\frac{\int_0^L EA(1 + M/mL - x/L)^2(1/L)^2\,dx}{\int_0^L m[(1 + M/mL)(x/L) - (x/L)^2/2]^2\,dx + M(1 + 2M/\text{mL})^2/4}$$

$$= \frac{\left(\frac{M}{mL}\right)^2 + \frac{M}{mL} + \frac{1}{3}}{\frac{1}{3}\left(\frac{M}{mL}\right)^2 + \frac{5}{12}\frac{M}{mL} + \frac{2}{15} + \frac{M}{mL}\left(\frac{1}{2} + \frac{M}{mL}\right)^2}\frac{EA}{mL^2} \tag{3.4.135}$$

3. $$\omega^2 = \frac{\int_0^L EA\left(\frac{\pi}{2L}\right)^2\cos^2\frac{\pi x}{2L}\,dx}{\int_0^L m\sin^2\frac{\pi x}{2L}\,dx + M} = \frac{\pi^2}{8\left(\frac{1}{2} + \frac{M}{mL}\right)}\frac{EA}{mL^2}$$

Table 3.4.9 Potential Energy for Various Systems

System	Potential energy* $V(t)$
Rods (also shafts and strings)	$\frac{1}{2}\int_0^L EA(x)[\partial u(x, t)/\partial x]^2 dx$
Beams	$\frac{1}{2}\int_0^L EI(x)[\partial^2 w(x, t)/\partial x^2]^2 dx$
Beams with axial force	$\frac{1}{2}\int_0^L \{EI(x)[\partial^2 w(x, t)/\partial x^2]^2 + P(x)[\partial\,\omega(x, t)/\partial x]^2\}dx$
Membranes	$\frac{1}{2}\int_{\text{Area}} T\{[\partial w(x, y, t)/\partial x]^2 + [\partial w(x, y, t)/\partial y]^2\}dx\,dy$
Plates	$\frac{1}{2}\int_{\text{Area}} D\{\nabla^2 w(x, y, t))^2 + 2(1 - \nu)[\partial^2 w(x, y, t)/\partial x\,\partial y\}^2$ $- (\partial^2 w(x, y, t)/\partial x^2)(\partial^2 w(x, y, t)/\partial y^2)]\}dx\,dy$

* If the distributed system has a spring at the boundary point a, then add a term $kw^2(a, t)/2$.

For comparison purposes, let $M = mL$, which yields the following estimates for the lowest natural frequency:

$$\begin{aligned} &1. \quad \omega = 0.8660\sqrt{\frac{EA}{mL^2}} \\ &2. \quad \omega = 0.8629\sqrt{\frac{EA}{mL^2}} \qquad (3.4.136) \\ &2. \quad \omega = 0.9069\sqrt{\frac{EA}{mL^2}} \end{aligned}$$

The best estimate is the lowest one, which corresponds to case 2, with the trial function in the form of the static displacement. Note that the estimate obtained in case 1 is also quite good. It corresponds to the first case in Table 3.4.2, representing a mass-spring system in which the mass of the spring is included.

Rayleigh-Ritz Method **Rayleigh's quotient**, Eq. (3.4.133), corresponding to any trial function $W(x)$ is always larger than the lowest eigenvalue ω_1^2, and it takes the minimum value of ω_1^2 when $W(x)$ coincides with the lowest natural mode $W_1(x)$. However, this possibility must be ruled out by virtue of the assumption that W_1 is not available. The **Rayleigh-Ritz method** is a procedure for minimizing Rayleigh's quotient by means of a sequence of approximate solutions converging to the actual solution of the eigenvalue problem. The minimizing sequence has the form

$$\begin{aligned} W(x) &= a_1\phi_i(x) \\ W(x) &= a_1\phi_1(x) + a_2\phi_2(x) = \sum_{j=1}^{2} a_j\phi_j(x) \qquad (3.4.137) \\ &\cdots \\ W(x) &= a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_n\phi_n(x) = \sum_{j=1}^{n} a_j\phi_j(x) \end{aligned}$$

where a_j are undetermined coefficients and $\phi_j(x)$ are suitable trial functions satisfying all, or at least the geometric boundary conditions. The coefficients $a_j (j = 1, 2, \ldots, n)$ are determined so that Rayleigh's quotient has a minimum. With Eqs. (3.4.137) inserted into Eq. (3.4.133), Rayleigh's quotient becomes

$$\Omega^2 = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}a_ia_j}{\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}a_ia_j} \qquad n = 1, 2, \ldots \qquad (3.4.138)$$

where $k_{ij} = k_{ji}$ and $m_{ij} = m_{ji}$ $(i, j = 1, 2, \ldots, n)$ are symmetric stiffness and mass coefficients whose nature depends on the potential energy and kinetic energy, respectively. The special case in which $n = 1$ represents Rayleigh's energy method. For $n \geq 2$, minimization of Rayleigh's quotient leads to the solution of the eigenvalue problem

$$\sum_{j=1}^{n} k_{ij}a_j = \Omega^2 \sum_{j=1}^{n} m_{ij}a_j \qquad i = 1, 2, \ldots, n; \; n = 2, 3, \ldots \qquad (3.4.139)$$

Equations (3.4.139) can be written in the matrix form

$$K\mathbf{a} = \Omega^2 M\mathbf{a} \qquad (3.4.140)$$

in which $K = [k_{ij}]$ is the symmetric stiffness matrix and $M = [m_{ij}]$ is the symmetric mass matrix. Equation (3.4.140) resembles the eigenvalue problem for multi-degree-of-freedom systems, Eq. (3.4.76), and its solutions possess the same properties. The eigenvalues Ω_r^2 provide approximations to the actual eigenvalues ω_r^2, and approach them from above as n increases. Moreover, the eigenvectors $\mathbf{a}_r = [a_{r1} \; a_{r2} \; \ldots \; a_{rn}]^T$ can be used to obtain the approximate natural modes by writing

$$W_r(x) = a_{r1}\phi_1(x) + a_{r2}\phi_2(x) + \cdots + a_{rn}\phi_n(x) = \sum_{j=1}^{n} a_{rj}\phi_j(x) \qquad r = 1, 2, \ldots, n; \; n = 2, 3, \ldots \qquad (3.4.141)$$

As an illustration, consider the same rod in axial vibration used to demonstrate Rayleigh's energy method. Insert Eqs. (3.4.137) with $W(x)$ replaced by $U(x)$ into the numerator and denominator of Eq. (3.4.134) to obtain

$$\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_0^L EA(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}\,dx\right) a_ia_j \qquad (3.4.142a)$$

$$\int_0^L m(x)U^2(x)\,dx + MU^2(L) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_0^L m(x)\phi_i(x)\phi_j(x)\,dx + M\phi_i(L)\phi_j(L)\right] a_ia_j \qquad (3.4.142b)$$

so that the stiffness and mass coefficients are

$$k_{ij} = \int_0^L EA(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}\,dx \qquad i, j = 1, 2, \ldots, n \qquad (3.4.143a)$$

$$m_{ij} = \int_0^L m(x)\phi_i(x)\phi_j(x)\,dx + M\phi_i(L)\phi_j(L) \qquad i, j = 1, 2, \ldots, n \qquad (3.4.143b)$$

respectively. As trial functions, use

$$\phi_j(x) = (x/L)^j \qquad j = 1, 2, \ldots, n \qquad (3.4.144)$$

which are zero at $x = 0$, thus satisfying the geometric boundary condition. Hence, the stiffness and mass coefficients are

$$k_{ij} = \frac{EAij}{L^{i+j}}\int_0^L x^{i-1}x^{j-1}\,dx = \frac{ij}{i+j-1}\,\frac{EA}{L} \qquad i, j = 1, 2, \ldots, n \qquad (3.4.145a)$$

$$m_{ij} = \frac{m}{L^{i+j}}\int_0^L x^ix^j\,dx + M = \frac{mL}{i+j+1} + M \qquad i, j = 1, 2, \ldots, n \qquad (3.4.145b)$$

so that the stiffness and mass matrices are

$$K = \frac{EA}{L}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 4/3 & 3/2 & \cdots & 2n/(n+1) \\ 1 & 3/2 & 9/5 & \cdots & 3n/(n+2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2n/(n+1) & 3n/(n+2) & \cdots & n^2/(2n-1) \end{bmatrix} \qquad (3.4.146a)$$

$$M = mL\begin{bmatrix} 1/3 & 1/4 & 1/5 & \cdots & 1/(n+2) \\ 1/4 & 1/5 & 1/6 & \cdots & 1/(n+3) \\ 1/5 & 1/6 & 1/7 & \cdots & 1/(n+4) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/(n+2) & 1/(n+3) & 1/(n+4) & \cdots & 1/(2n+1) \end{bmatrix} + M\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \qquad (3.4.146b)$$

For comparison purposes, consider the case in which $M = mL$. Then, for $n = 2$, the eigenvalue problem is

$$\begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda\begin{bmatrix} 4/3 & 5/4 \\ 5/4 & 6/5 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \lambda = \Omega^2\frac{mL^2}{EA} \qquad (3.4.147)$$

which has the solutions

$$\begin{aligned} \lambda_1 &= 0.7407 \qquad \mathbf{a}_1 = [1 \quad -0.1667]^T \\ \lambda_2 &= 12.0000 \qquad \mathbf{a}_2 = [1 \quad -1.0976]^T \end{aligned} \qquad (3.4.148)$$

Hence, the computed natural frequencies and modes are

$$\Omega_1 = 0.8607\sqrt{\frac{EA}{mL^2}} \qquad U_1(x) = \frac{x}{L} - 0.1667\left(\frac{x}{L}\right)^2$$
$$\Omega_2 = 3.4641\sqrt{\frac{EA}{mL^2}} \qquad U_2(x) = \frac{x}{L} - 1.0976\left(\frac{x}{L}\right)^2 \qquad (3.4.149)$$

Comparing Eqs. (3.4.149) with the estimates obtained by Rayleigh's energy method, Eqs. (3.4.136), note that the Rayleigh-Ritz method has produced a more accurate approximation for the lowest natural frequency. In addition, it has produced a first approximation for the second lowest natural frequency, as well as approximations for the two lowest modes, which Rayleigh's energy method is unable to produce. The approximate solutions can be improved by letting $n = 3, 4, \ldots$.

Finite Element Method In the Rayleigh-Ritz method, the trial functions extend over the entire domain of the system and tend to be complicated and difficult to work with. More importantly, they often cannot be produced, particularly for two-dimensional problems. Another version of the Rayleigh-Ritz method, the **finite element method,** does not suffer from these drawbacks. Indeed, the trial functions extending only over small subdomains, referred to as **finite elements,** are known low-degree polynomials and permit easy computer coding. As in the Rayleigh-Ritz method, a solution is assumed in the form of a linear combination of trial functions, known as **interpolation functions,** multiplied by undetermined coefficients. In the finite element method the coefficients have physical meaning, as they represent ''nodal'' displacements, where ''nodes'' are boundary points between finite elements. The computation of the stiffness and mass matrices is carried out for each of the elements separately and then the element stiffness and mass matrices are assembled into global stiffness and mass matrices. One disadvantage of the finite element method is that it requires a large number of degrees of freedom.

To illustrate the method, and for easy visualization, consider the transverse vibration of a string fixed at $x = 0$ and with a spring of stiffness K attached at $x = L$ (Fig. 3.4.27) and divide the length L into n elements of width h, so that $nh = L$. Denote the displacements of the nodal points x_e by a_e and assume that the string displacement is linear between any two nodal points. Figure 3.4.28 shows a typical element e.

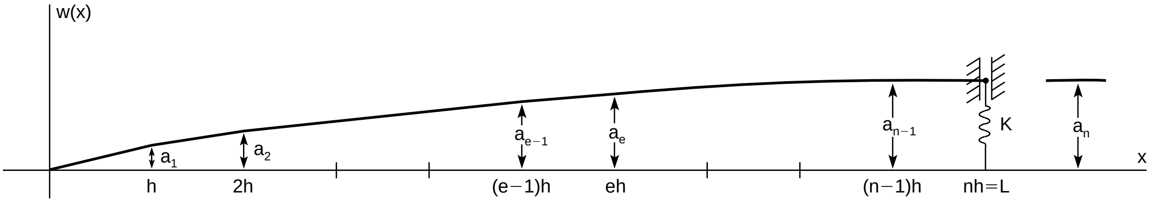


Fig. 3.4.27

The process can be simplified greatly by introducing the nondimensional local coordinate $\xi = j - x/h$. Then, considering the two linear interpolation functions

$$\phi_1(\xi) = \xi \qquad \phi_2(\xi) = 1 - \xi \qquad (3.4.150)$$

the displacement at point ξ can be expressed as

$$\omega(\xi) = a_{e-1}\phi_1(\xi) + a_e\phi_2(\xi) \qquad (3.4.151)$$

where a_{e-1} and a_e are the nodal displacements for element e. Using Eqs. (3.4.143) and changing variables from x to ξ, we can write the element stiffness and mass coefficients

$$k_{eij} = \frac{1}{h}\int_0^1 EA\,\frac{d\phi_i}{d\xi}\frac{d\phi_j}{d\xi}\,d\xi \qquad m_{eij} = h\int_0^1 m\phi_i\phi_j d\xi, \qquad i, j = 1, 2 \qquad (3.4.152)$$

Introducing Eq. (3.4.151) into Eqs. (3.4.152) and considering the boundary conditions, we obtain the element stiffness and mass matrices

$$K_1 = \frac{EA}{h} \qquad K_e = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad e = 2, 3, \ldots, n-1$$
$$K_n = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & Kh/EA \end{bmatrix} \qquad M_1 = \frac{hm}{3}$$
$$M_e = \frac{hm}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad e = 2, 3, \ldots, n \qquad (3.4.153)$$

where K_1 and M_1 are really scalars, because the left end of the first element is fixed, so that the displacement is zero. Then, since the nodal displacement a_e is shared by elements e and $e + 1$ ($e = 1, 2, \ldots, n - 2$), the element stiffness and mass matrices can be assembled into the global stiffness and mass matrices

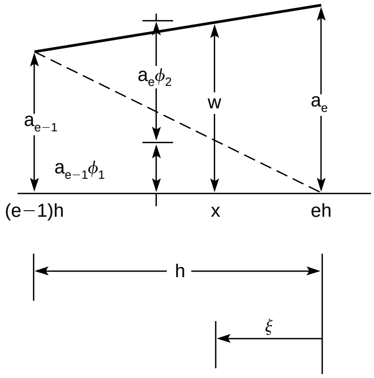


Fig. 3.4.28

$$K = \frac{EA}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & Kh/EA \end{bmatrix} \qquad (3.4.154)$$

$$M = \frac{hm}{6}\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}$$

For beams in bending, the displacements consist of one translation and one rotation per node; the interpolation functions are the Hermite cubics

$$\begin{aligned}\phi_1(\xi) &= 3\xi^2 - 2\xi^3,\ \phi_2(\xi) = \xi^2 - \xi^3\\ \phi_3(\xi) &= 1 - 3\xi^2 + 2\xi^3,\ \phi_4(\xi) = -\xi + 2\xi^2 - \xi^3\end{aligned} \tag{3.4.155}$$

and the element stiffness and mass coefficients are

$$k_{eij} = \frac{1}{h^3}\int_0^1 EI\,\frac{d^2\phi_i}{d\xi^2}\,\frac{d^2\phi_j}{d\xi^2}\,d\xi \qquad m_{eij} = h\int_0^1 m\phi_i\phi_j d\xi$$

$$i, j = 1, 2, 3, 4 \tag{3.4.156}$$

yielding typical element stiffness and mass matrices

$$K_e = \frac{EI}{h^3}\begin{bmatrix} 12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}$$

$$M_e = \frac{hm}{420}\begin{bmatrix} 156 & 22 & 54 & -13\\ 22 & 4 & 13 & -3\\ 54 & 13 & 156 & -22\\ -13 & -3 & -22 & 4\end{bmatrix} \tag{3.4.157}$$

The treatment of two-dimensional problems, such as for membranes and plates, is considerably more complex (see Meirovitch, ‘‘Principles and Techniques of Vibration,’’ Prentice-Hall) than for one-dimensional problems.

The various steps involved in the finite element method lend themselves to ready computer programming. There are many computer codes available commercially; one widely used is NASTRAN.

VIBRATION-MEASURING INSTRUMENTS

Typical quantities to be measured include acceleration, velocity, displacement, frequency, damping, and stress. Vibration implies motion, so that there is a great deal of interest in transducers capable of measuring motion relative to the inertial space. The basic transducer of many vibration-measuring instruments is a mass-damper-spring enclosed in a case together with a device, generally electrical, for measuring the displacement of the mass relative to the case, as shown in Fig. 3.4.29. The equation for the displacement $z(t)$ of the mass relative to the case is

$$m\ddot{z}(t) + c\dot{z}(t) + kz(t) = -m\ddot{y}(t) \tag{3.4.158}$$

where $y(t)$ is the displacement of the case relative to the inertial space. If this displacement is harmonic, $y(t) = Y \sin \omega t$, then by analogy with Eq. (3.4.35) the response is

$$\begin{aligned} z(t) &= Y\left(\frac{\omega}{\omega_n}\right)^2 |G(\omega)| \sin(\omega t - \phi)\\ &= Z(\omega)\sin(\omega t - \phi) \end{aligned} \tag{3.4.159}$$

so that the magnitude factor $Z(\omega)/Y = (\omega/\omega_n)^2\,|G(\omega)|$ is as plotted in Fig. 3.4.9 and the phase angle ϕ is as in Fig. 3.4.4. The plot $Z(\omega)/Y$ versus ω/ω_n is shown again in Fig. 3.4.30 on a scale more suited to our purposes.

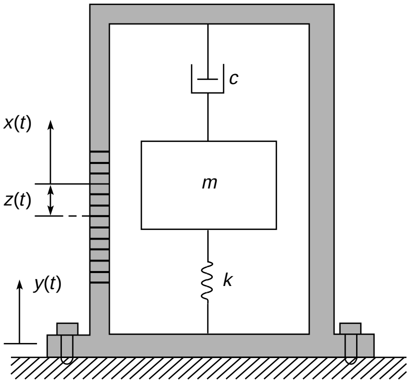


Fig. 3.4.29

Accelerometers are high-natural-frequency instruments. Their usefulness is limited to a frequency range well below resonance. Indeed, for small values of ω/ω_n, Eq. (3.4.159) yields the approximation

$$Z(\omega) \approx \frac{1}{\omega_n^2}\,\omega^2 Y \tag{3.4.160}$$

so that the signal amplitude is proportional to the amplitude of the acceleration of the case relative to the inertial space. For $\zeta = 0.7$, the accelerometer can be used in the range $0 \le \omega/\omega_n \le 0.4$ with less than 1 percent error, and the range can be extended to $\omega/\omega_n \le 0.7$ if proper corrections, based on instrument calibration, are made.

Commonly used accelerometers are the compression-type piezoelectric accelerometers. They consist of a mass resting on a piezoelectric ceramic crystal, such as quartz, tourmaline, or ferroelectric ceramic, with the crystal acting both as spring and sensor. Piezoelectric actuators have negligible damping, so that their range must be smaller, such as $0 < \omega/\omega_n < 0.2$. In view of the fact, however, that the natural frequency is very high, about 30,000 Hz, this is a respectable range.

Displacement-Measuring Instruments These are low-natural-frequency devices and their usefulness is limited to a frequency range well above resonance. For $\omega/\omega_n \gg 1$, Eq. (3.4.159) yields the approximation

$$Z(\omega) \approx Y \tag{3.4.161}$$

so that the signal amplitude is proportional to the amplitude of the case displacement. Instruments with low natural frequency compared to the excitation frequency are known as **seismometers**. They are commonly used to measure ground motions, such as those caused by earthquakes or underground nuclear explosions. The requirement of low natural frequency dictates that the mass, referred to as **seismic mass,** be very large and the spring very soft, so that essentially the mass remains

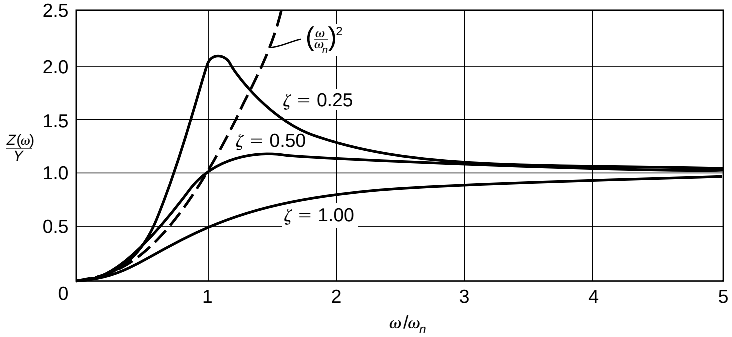


Fig. 3.4.30

stationary in an inertial space while the case attached to the ground moves relative to the mass.

Seismometers tend to be considerably larger in size than accelerometers. If a large-size instrument is undesirable, or even if size is not an issue, displacements in harmonic motion, as well as velocities, can be obtained from accelerometer signals by means of electronic integrators.

Some other transducers, not mass-damper-spring transducers, are as follows (Harris, ‘‘Shock and Vibration Handbook,’’ 3d ed., McGraw-Hill):

Differential-transformer pickups: They consist of a core of magnetic material attached to the vibrating structure, a primary coil, and two secondary coils. As the core moves, both the inductance and induced voltage of one secondary coil increase while those of the other decrease. The output voltage is proportional to the displacement over a wide range. Such pickups are used for very low frequencies, up to 60 Hz.

Strain gages: They consist of a grid of fine wires which exhibit a change in electrical resistance proportional to the strain experienced. Their flimsiness requires that strain gages be either mounted on or bonded to some carrier material.