Contemporary Mathematics

Some presentations of the trivial group

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Abstract. We exhibit a large family of two generator, two relator presentations of the trivial group: if w is a word in a and b with exponent sum 0 on a and $n > 0$, then $< a, b \mid a^{-1}b^n a = b^{n+1}, a = w >$ is a presentation of the trivial group.

For topological reasons, balanced presentations (those having the same number of generators as relators) of the trivial group are of particular interest. Andrews and Curtis [**1**],[**2**] conjectured that a balanced presentation of the trivial group can be transformed by certain restricted moves to the presentation whose relators are just the generators. See the article by Burns and Macedonska [**3**] for a survey. The topological interpretation of the Andrews-Curtis conjecture is discussed by Wright in [**5**].

One well known example of a balanced presentation of the trivial group is $< a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 >$. A search for more examples led us to ask which two relator presentations of the form $< a, b \mid a^{-1}b^n a = b^{n+1}, a = w >$, where w is a word in a and b, are presentations of the trivial group. A necessary condition is that the group presented must be perfect, so we may as well assume that w has exponent sum 0 on the generator a.

George Havas generously made available his coset enumeration programs for us to experiment on examples of this form (see [**4**]). Using progressively more complicated words w which required progressively larger and faster computers to complete the enumeration, we always obtained the same result: the group was trivial. Encouraged by this experimental evidence we eventually saw how to prove the following theorem. [1]

Theorem 1. *Let G be a perfect group generated by two elements a and b which satisfy the equation $a^{-1}b^n a = b^m$ where $n > 0$ and $m > 0$ are relatively prime. Then G is the trivial group.*

The theorem follows easily from the following lemma which contains the essence of the argument.

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[1]We found this family of presentations some years ago. Recently colleagues involved in computational group theory expressed interest and urged us to publish our result. We thank Geoff Smith and George Havas for corrections and suggestions.

Lemma 2. *Let $m > n > 0$ be relatively prime and suppose that H is the group with presentation*

$$H =< a, b \mid a^{-1}b^n a = b^m, a = w >$$

where w is a word on a and b having exponent sum 0 on a. Then $a =_H b^q$, where q is the exponent sum on b in w, and H is the cyclic group of order $m - n$ generated by b.

Proof. From the relation $a^{-1}b^n a = b^m$ it follows that $a^{-1}b^{nk}a = b^{mk}$ and $ab^{mk}a^{-1} = b^{nk}$ for any integer k. In particular we have

$$a^{-1}b^{n^i m^j}a = b^{n^{i-1}m^{j+1}} \text{ and } ab^{n^i m^j}a^{-1} = b^{n^{i+1}m^{j-1}}$$

provided $i > 0, j \geq 0$ in the first case and $i \geq 0, j > 0$ in the second. Thus conjugation of $b^{n^i m^j}$ by a or a^{-1} simply shifts the exponents of n and m in the power of b by ± 1.

Let k denote the total number of occurrences of a and a^{-1} in the word w. Then since w has exponent sum 0 on a it follows from the above relations that

$$w^{-1}b^{n^k m^k}w = b^{n^k m^k}$$

because the net effect of the conjugations by a and a^{-1} (shifting backwards and forwards) is to leave this power of b fixed. Hence w commutes with $b^{n^k m^k}$.

Now $a =_H w$ so $b^{n^k m^k} = a^{-1}b^{n^k m^k}a = b^{n^{k-1}m^{k+1}}$ and thus $b^{n^{k-1}m^k(n-m)} =_H 1$. Hence also

$$b^{n^{2k-1}(n-m)} = a^k b^{n^{k-1}m^k(n-m)}a^{-k} =_H 1$$

and

$$b^{m^{2k-1}(n-m)} = a^{-(k-1)}b^{n^{k-1}m^k(n-m)}a^{k-1} =_H 1.$$

Thus the order of b divides $n^{2k-1}(n-m)$ and $m^{2k-1}(n-m)$. Since n and m are relatively prime, it follows that the order of b divides $n - m$ and is relatively prime to both n and m. In particular, $b^{(n-m)} = 1$ so $b^n = b^m$. If B is the cyclic subgroup generated by b, then $b^n = b^m$ generates B since n and m are prime to the order of b. Since a and b generate H and $a^{-1}b^n a = b^m$, it follows that B is central. From $a = w$ it follows that $a =_H b^q$ where q is the exponent sum on b in w. Sincs $b^{m-n} = 1$, H is the cyclic group of order $m - n$ generated by b. This completes the proof of the lemma.

Proof of the theorem : Changing notation if necessary, we may assume that $m > n > 0$. Since G is perfect and a and b generate G, there must be an equation of the form $a =_G w$ which holds in G where w is a word in a and b having exponent sum 0 in both a and b. Then G is a quotient group of the group H described in the lemma, and so G is cyclic. But G is also perfect, so G must be the trivial group. This proves the theorem.

We note that either the theorem or the lemma immediately yields the following large family of balanced presentations of the trivial group discussed above:

Corollary 3. *Let w be any word in a and b and their inverses which has exponent sum 0 on a. If $n > 0$ then the balanced presentation*

$$< a, b \mid a^{-1}b^n a = b^{n+1}, a = w >$$

is a presentation of the trivial group.

Greg Conner has pointed out to us the following consequence of the above theorem.

Corollary 4. *The group* $< a, b \mid a^{-1}b^n a = b^m >$ *where* $m, n > 0$ *has a non-trivial perfect quotient group if and only if* $\gcd(m, n) > 1$.

One direction is immediate. For the other, we observe that if $p > 1$ is a prime divisor of both m and n, then adding the relation $b^p = 1$ yields as quotient group the ordinary free product of the infinite cycle on a and the cyclic group of order p generated by b. This free product is easily seen to have a non-trivial perfect quotient.

References

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